CSE 16 4-30-24

Summation Formulas ... continue ...

(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ Proof later

(3) $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ " "

(4) $\sum_{k=0}^{n} ar^k = \begin{cases} a(n+1) & \text{if } r=1 \\ a\left(\frac{r^{n+1}-1}{r-1}\right) & \text{if } r \neq 1 \end{cases}$

Proof of (4) $(r \neq 1)$

$$S = \sum_{k=0}^{n} ar^k$$

$$rS = \quad \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^n} + ar^{n+1}$$
$$S = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} + \cancel{ar^n}$$

$$rS - S = ar^{n+1} - a$$

$$(r-1)S = a(r^{n+1} - 1)$$

$$\therefore \quad S = a\left(\frac{r^{n+1} - 1}{r - 1}\right)$$

Ex. $$\sum_{k=5}^{10} 3\cdot 2^k$$

$$= \sum_{k=0}^{10} 3\cdot 2^k - \sum_{k=0}^{4} 3\cdot 2^k$$

$$= 3\cdot\left(\frac{2^{11}-1}{2-1}\right) - 3\left(\frac{2^5-1}{2-1}\right)$$

$$= 3\left(2^{11}-2^5\right) = \boxed{6048}$$

Ex. Prior example

$$(a_n - a_{n-1}) = 3(a_{n-1} - a_{n-2})$$

let $b_n = a_n - a_{n-1}$. Then

$$\begin{cases} b_n = 3b_{n-1} \\ b_0 = 2 \end{cases} \longrightarrow \boxed{b_n = 2\cdot 3^n}$$

so, the original seq.

$$a_n - a_{n-1} = b_n = 2 \cdot 3^n$$

$$\therefore \quad a_n = a_{n-1} + 2 \cdot 3^n$$

$$a_1 = 1 + 2 \cdot 3$$

$$a_2 = 1 + 2 \cdot 3 + 2 \cdot 3^2$$

$$a_3 = 1 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3$$

$$\vdots$$

$$a_n = 1 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots\cdots + 2 \cdot 3^n$$

$$= 1 - 2 \cdot 3^0 + \left(2 \cdot 3^0 + 2 \cdot 3^1 + \cdots + 2 \cdot 3^n\right)$$

$$= -1 + \sum_{k=0}^{n} 2 \cdot 3^k$$

$$= -1 + \cancel{2}\left(\frac{3^{n+1} - 1}{\cancel{3-1}}\right)$$

$$\boxed{a_n = -2 + 3^{n+1}}$$

2.5 Cardinality of sets

<u>Defn</u>

Two sets A, B (not necessarily finite) are said to have the same <u>cardinality</u> iff there exists a bijection

$$f : A \to B$$

<u>notation</u>: $|A| = |B|$

Ex. $|\mathbb{N}| = |\mathbb{Z}^+|$

$$\begin{array}{cccccccc} \mathbb{N} = \{ & 0, & 1, & 2, & 3, & \ldots\ldots & \} \\ & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbb{Z}^+ = \{ & 1, & 2, & 3, & 4, & \ldots\ldots & \} \end{array}$$

$$f : \mathbb{N} \to \mathbb{Z}^+, \quad f(n) = n+1$$

Ex. $E = \{\text{even \#s in } \mathbb{N}\} = \{0, 2, 4, \ldots\}$

$O = \{\text{odd \#s in } \mathbb{N}\} = \{1, 3, 5, \ldots\}$

$|\mathbb{N}| = |E| \ : \ f(n) = 2n$

$|\mathbb{N}| = |O| \ : \ f(n) = 2n+1$

Ex. $|\mathbb{N}| = |\mathbb{Z}|$

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$\mathbb{N} \xrightarrow{f} \mathbb{Z}$$

0	0
1	1
2	-1
3	2
4	-2
⋮	⋮

Exercise

Prove that if $|A|=|B|$ and $|B|=|C|$, then $|A|=|C|$.

Defn

A set S is called countable iff either

(1) S is finite ($|S|<\infty$).

or

(2) $|S|=|\mathbb{N}|$ (or $|\mathbb{Z}^+|$, or $|\mathbb{Z}|$, ...)

In case (2) we say S is <u>countably infinite</u>.

If S is not countable, its called <u>uncountable</u>

Informally: a set is countable iff its elements can form a list (finite or infinite)

S	$\mathbb{Z}^+$
x_1	1
x_2	2
x_3	3
$\vdots$	$\vdots$

Ex $\mathbb{Z}^+ \times \mathbb{Z}^+$

$$
\begin{array}{ccccc}
11 & 12 & 13 & 14 & \cdots \\
21 & 22 & 23 & 24 & \cdots \\
31 & 32 & 33 & 34 & \cdots \\
41 & 42 & 43 & 44 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{Z}^+ & & \mathbb{Z}^+ \times \mathbb{Z}^+ \\
1 & \longrightarrow & 11 \\
2 & \longrightarrow & 21 \\
3 & \longrightarrow & 12 \\
4 & \longrightarrow & 31 \\
5 & \longrightarrow & 22 \\
6 & \longrightarrow & 13 \\
\vdots & & \vdots
\end{array}
$$

Exercise

find an explicit formula for f

$$f : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$$

(see 2.5 #31, p. 177)

Exercise

Show that any subset of a countable set is countable.

(Hint: its sufficient to show any subset of $\mathbb{Z}^+$ is countable.)

Exercise

Show $\mathbb{Q}$ is countable. (Hint: find a bijection from $\mathbb{Q}^+$ to a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$.)

Theorem

$\mathbb{R}$ is uncountable.

Proof (contradiction)

Assume $\mathbb{R}$ is countable. Then so is

$$S = (0,1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$$

since S is countable, it is the range of a sequence

$$S = \{r_1, r_2, r_3, r_4, \ldots\ldots\}$$

each element of S has a <u>unique</u>* decimal expansion

$$\begin{array}{l} r_1 = .d_{11} d_{12} d_{13} \cdots\cdots \\ r_2 = .d_{21} d_{22} d_{23} \cdots\cdots \\ r_3 = .d_{31} d_{32} d_{33} \cdots\cdots \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots \end{array}$$

where each $d_{ij} \in \{0, 1, 2, \ldots, 9\}$

* if a number has 2 decimal expansions, choose the one with tail all 9's, not all 0's.

$$.4999\ldots\ldots = \cancel{.5000\ldots\ldots}$$

Define $x \in S$ by

$$x = .c_1 c_2 c_3 \ldots\ldots$$

where

$$c_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

Observe $c_i \neq d_{ii}$ for all $i = 1, 2, 3, \ldots$

hence $x \neq r_i$ for any $i = 1, 2, 3 \ldots$

Therefore

$$x \notin S$$

But $x \in S$ by its very construction.

This contradiction shows our assumption was false.

$\therefore$ $\mathbb{R}$ is uncountable. $\blacksquare$

<u>Rmks</u>

- $|\mathbb{N}| \neq |\mathbb{R}|$
- called 'Cantor diagonal argument'

<u>Theorem</u>

If $A_1, A_2, \ldots$ is a sequence of countable sets, then

$$S = \bigcup_{n=1}^{\infty} A_n$$

is also countable

<u>Exercise</u> Prove this

Hint! Similar to proof that

$$|\mathbb{Z}^+| = |\mathbb{Z}^+ \times \mathbb{Z}^+|$$

Theorem

Let S be a set. Then no function

$$f : S \to \mathbb{P}(S)$$

can be surjective.

Proof (contradiction)

Assume $f : S \to \mathbb{P}(S)$ is surjective. Define $A \subseteq S$

$$A = \{x \in S \mid x \notin f(x)\} \subseteq S$$

Since f is surjective, there exists $y \in S$ such that $f(y) = A$.

But

$$y \in A \rightarrow y \in f(y) \rightarrow y \notin A \quad ⨳$$

and

$$y \notin A \rightarrow y \notin f(y) \rightarrow y \in A \quad ⨳$$

$\therefore$ no such f exists. $\blacksquare$