CSE 16 4-25-24

Thm

$f: A \to B$ is invertible iff bijective

Proof

($\Rightarrow$) suppose f is invertible, i.e. f^{-1} exist ✓

- injective ✓
- surjective: ✓

let $y \in B$. Let $x = f^{-1}(y)$. Then

$$\begin{aligned} f(x) &= f(f^{-1}(y)) \\ &= f \circ f^{-1}(y) \\ &= i_B(y) \\ &= y. \end{aligned}$$

($\Leftarrow$) assume f is bijective.

Define $g : B \to A$ by

for each $y \in B$

$g(y) =$ the unique $x \in A$ such that $f(x) = y$

<u>Note</u>: Such an x exists because f is surjective. Also x is unique since f is injective.

Then necessarily

$$f(g(y)) = y \quad \text{for all } y \in B.$$

and

$$g(f(x)) = x \quad \text{for all } x \in A.$$

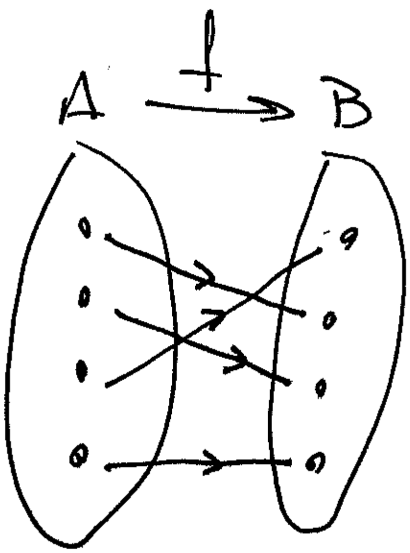
So

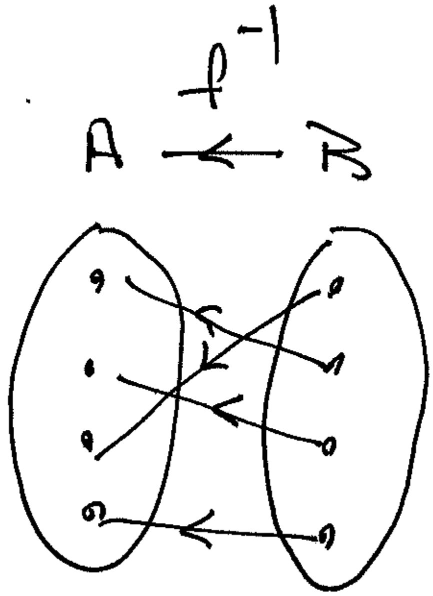
$$f \circ g = i_B \text{ and } g \circ f = i_A$$

$\therefore$ $g = f^{-1}$ and f is invertible.

$\blacksquare$

Picture





<u>Ex.</u> $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is bijective, so its invertibly

$$\boxed{g^{-1}(y) = \frac{y-3}{2}}$$

$$y = 2x+3$$

$$y-3 = 2x$$

$$x = \frac{y-3}{2}$$

<u>Ex.</u> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ not invertible.

<u>Ex</u> $f : [0,\infty) \to [0,\infty)$, $f(x) = x^2$ is invertible, and $f^{-1}(x) = \sqrt{x}$

Exercises

Prove

(a) if f is invertible, then so is f^{-1} and $(f^{-1})^{-1} = f$

(b) if f is bijective, then so is f^{-1}.

2.4 Sequences & Summations

Defn

A *sequence* is a function whose Domain is a subset of $\mathbb{Z}$, usually $\mathbb{N} = \{0, 1, 2, \ldots\}$ or $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Remarks

- write a_n instead of $a(n)$ for the image of n under a
- a_n is called the n^{th} term.

- Book writes $\{a_n\} = \{a_n\}_{n=1}^{\infty}$.

We write

$$(a_n) = (a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots)$$

$$(a_n)_{n=1}^{10} = (a_1, a_2, \ldots, a_{10})$$

$$a : \{1, \ldots, 10\} \to \,?$$

Ex

- $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$
- $(2^n)_{n=0}^{\infty} = (1, 2, 4, 8, 16, \ldots)$
- $(n^2)_{n=3}^{6} = (9, 16, 25, 36)$

Defn

A **geometric sequence** is of the form

$$(a \cdot r^n)_{n=0}^{\infty} = (a, ar, ar^2, ar^3, \dots)$$

initial term a common ratio r

Ex $(2 \cdot 3^n) = (2, 6, 18, 54, \dots)$

<u>Defn</u>

An <u>arithmetic sequence</u> is of the form

$$(a+nd)_{n=0}^{\infty} = (a, a+d, a+2d, a+3d, \ldots)$$

initial term: a

common difference: d

<u>Ex</u> $(-1+4n) = (-1, 3, 7, 11, \ldots)$

Defn

A <u>recurrence relation</u> (or <u>recurrence</u>) for (a_n) is an equation that expresses a_n in terms of earlier terms: $a_{n-1}, a_{n-2}, \ldots, a_0$

Ex. $\boxed{a_n = a_{n-1} + 3}$, $\boxed{a_0 = 2}$

defines $(2+3n)_{n=0}^{\infty} = (2, 5, 8, 11, \ldots)$

Ex. $\underbrace{\boxed{a_n = a_{n-1} - a_{n-2}}}_{\text{recurrence}}$, $\underbrace{\boxed{a_0 = 3, a_1 = 5}}_{\text{initial terms}}$

Ex. Fibonacci Sequence

$$F_n = F_{n-1} + F_{n-2}\,, \quad F_0 = 0\,, \quad F_1 = 1$$

$$(F_n) = (0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\ldots)$$

Ex. Lucas seq.

$$L_n = L_{n-1} + L_{n-2}\,, \quad L_0 = 2\,, \quad L_1 = 1$$

$$(L_n) = (2, 1, 3, 4, 7, 11, 18, \ldots\ldots)$$

Defn

A solution to a recurrence relation

$$(*) \qquad a_n = F(a_{n-1}, a_{n-2}, \ldots, a_0)$$

is a closed formula (a function of n only) satisfying $(*)$

$$a_n = f(n)$$

i.e. $f(n) = F(f(n-1), f(n-2), \ldots, f(0))$

for all $n \in \mathbb{N}$.

Ex.

$$\begin{cases} a_0 = 1 \\ a_n = n \cdot a_{n-1} \end{cases}$$

Solution: $\boxed{a_n = n!}$

Ex. $$\begin{cases} b_0 = 1 \\ b_n = 3 b_{n-1} \end{cases}$$

Solution: $b_n = 3^n$

Ex. $c_n = 5c_{n-1} - 6c_{n-2}$

check: ① $c_n = 2^n$ yes

② $c_n = 3^n$ yes

③ $c_n = 5^n$ no

① RHS $= 5c_{n-1} - 6c_{n-2}$

$= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2}$

$= 5\, 2^{n-1} - 3 \cdot 2^{n-1}$

$= 2^{n-1}(5-3)$

$= 2 \cdot 2^{n-1} = 2^n = c_n =$ LHS

② $RHS = 5C_{n-1} - 6C_{n-2}$

$$= 5 \cdot 3^{n-1} - 6 \cdot 3^{n-2}$$

$$= 5 \cdot 3^{n-1} - 2 \cdot 3^{n-1}$$

$$= (5-2) \cdot 3^{n-1}$$

$$= 3 \cdot 3^{n-1} = 3^n = C_n = LHS$$

③ $RHS = 5C_{n-1} - 6C_{n-2}$

$$= 5 \cdot 5^{n-1} - 6 \cdot 5^{n-2}$$

$$= (5 \cdot 5 - 6) 5^{n-2}$$

$$= 19 \cdot 5^{n-2}$$

$$LHS = C_n = 5^n \neq 19 \cdot 5^{n-2} = RHS$$

Exercise

- check

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Hint: let $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$

and show that ϕ and $\bar{\phi}$ are solutions to: $x^2 - x - 1 = 0$, i.e.

$x^2 = x + 1$, so

$$\phi^2 = \phi + 1$$
$$\bar{\phi}^2 = \bar{\phi} + 1$$

- check: $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

Ex. find a recurrence relation for the seq.

	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	....
	1,	7,	25,	79,	241,	727,	2185,	6559,	.....
diff:	6	18	54	162	486	1458	4374		
ratio:	3	3	3	3	3	3			

we see that

$$a_n - a_{n-1} = 3(a_{n-1} - a_{n-2})$$

$$\therefore \quad \begin{cases} a_n = 4a_{n-1} - 3a_{n-2} \\ a_0 = 1 \\ a_1 = 7 \end{cases}$$

check: $\boxed{a_n = 3^{n+1} - 2}$

Defn

A summation (or series) is the sum of a sequence. (finite or infinite.)

notation

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Ex. $\displaystyle\sum_{i=0}^{6} 2^i = 1 + 2 + 4 + 8 + 16 + 32 + 64$

Ex. $\displaystyle\sum_{k \in \{2,4,6\}} k^2 = 2^2 + 4^2 + 6^2$

Special formulas

(1) $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

Thm: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Proof. let

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$S + S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$\therefore \; 2S = n(n+1)$

$\therefore \; S = \frac{n(n+1)}{2}$ ▨