cse 16 4-23-24

Notation

Given $A_1, A_2, A_3, \ldots\ldots$

- $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$
- $A_1 \cup A_2 \cup \cdots\cdots = \bigcup_{i=1}^{\infty} A_i$
- $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$
- $A_1 \cap A_2 \cap \cdots\cdots = \bigcap_{i=1}^{\infty} A_i$

2.3 Functions

Defn

A function (or map, mapping, transformation) consists of

(1) a set A called Domain

(2) a set B called Codomain

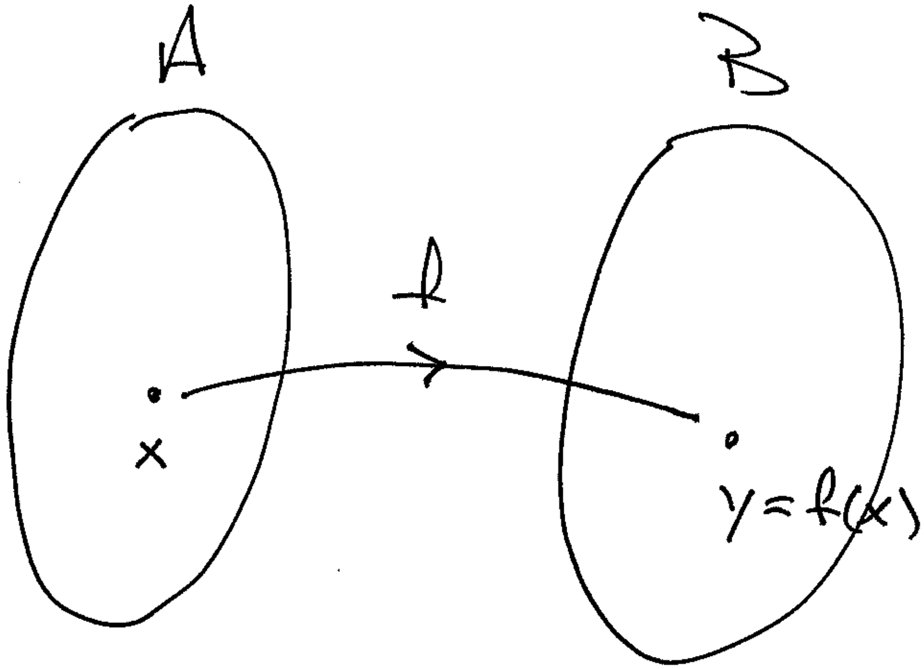
(3) a rule f that assigns to each $x \in A$ a unique $y \in B$.

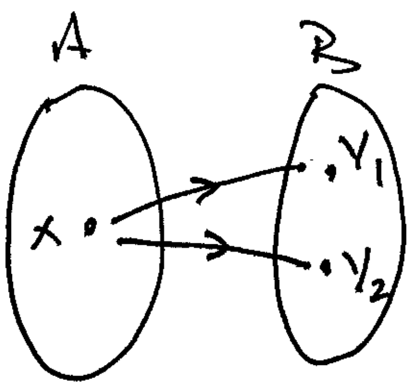
notation: $y = f(x)$

call: y the image of x under f

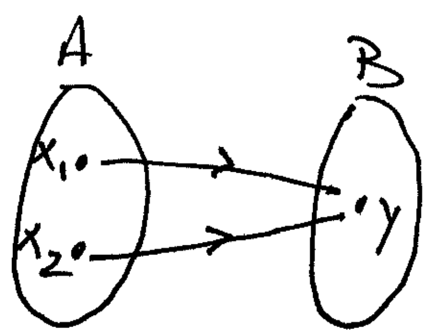
call: x a preimage of y under f

notation: $f : A \to B$





not a function

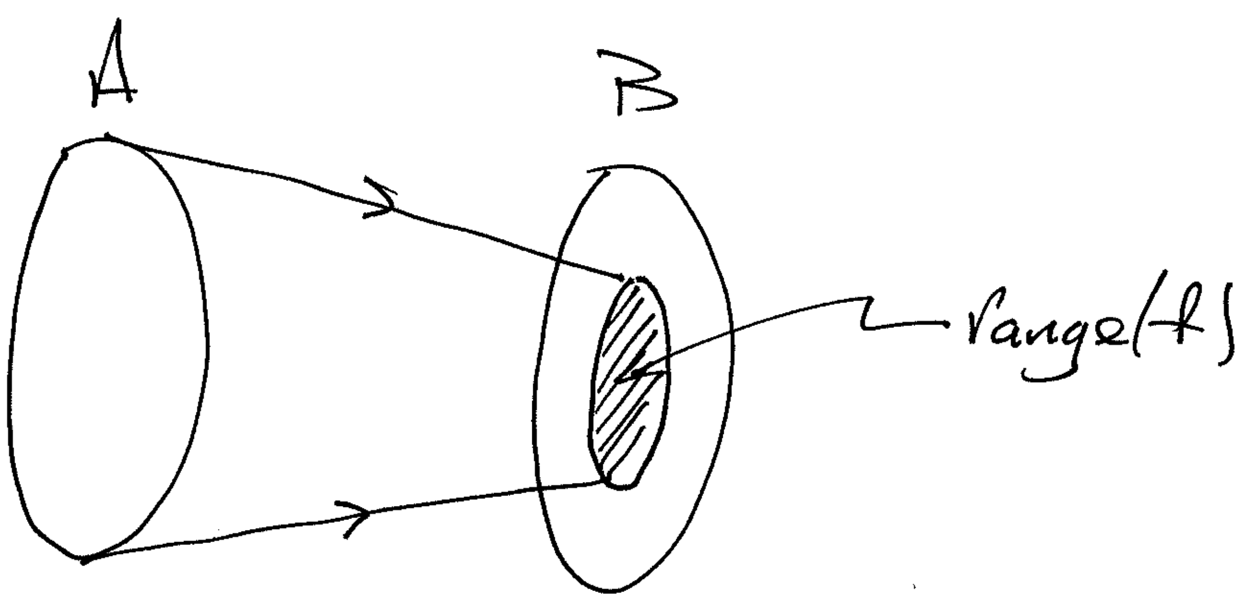


OK, is a function

Defn

The <u>range</u> of f is the set

$$\text{range}(f) = \{y \in B \mid \exists x \in A : y = f(x)\} \subseteq B$$



Ex. $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

$$\text{range}(f) = \{0, 1, 4, 9, 16, 25, 36, \ldots\}$$

Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$\text{range}(f) = [0, \infty)$$
$$= \{ y \in \mathbb{R} \mid y \geq 0 \}$$

Defn

The image of $S \subseteq A$ under f is

$$f(S) = \{ y \in B \mid \exists x \in S : y = f(x) \}$$
$$\subseteq \text{range}(f) \subseteq B$$

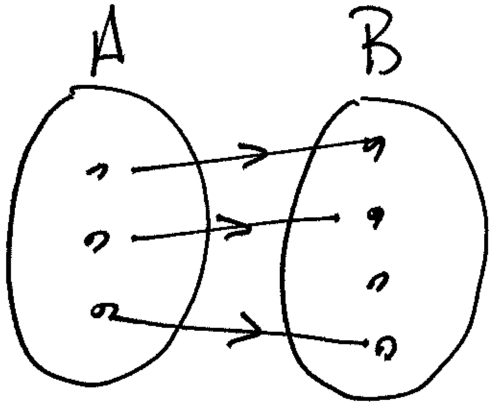
Defn

A function $f: A \to B$ is called injective (also one-to-one) iff

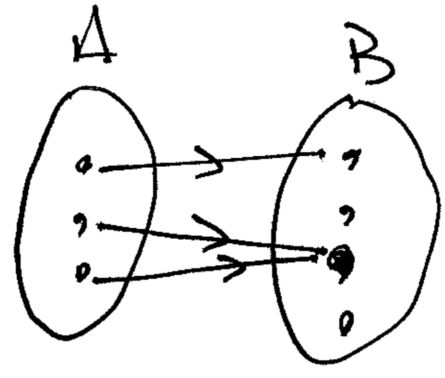
$$\forall x_1 \forall x_2 : f(x_1) = f(x_2) \to x_1 = x_2$$

equivalently

$$\forall x_1 \forall x_2 : x_1 \neq x_2 \to f(x_1) \neq f(x_2)$$



injective



not injective

Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ not injective

Ex. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is injective

show $\forall x_1 \forall x_2 : g(x_1) = g(x_2) \to x_1 = x_2$

Proof.

Let $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. Assume

$$2x_1 + 3 = 2x_2 + 3$$

$$\therefore \quad 2x_1 = 2x_2$$

$$\therefore \quad x_1 = x_2$$

$\blacksquare$

<u>Defn</u>

We say $f : \mathbb{R} \to \mathbb{R}$ is <u>strictly increasing</u> (<u>strictly decreasing</u>)

iff $x < y \to f(x) < f(y)$

(iff $x < y \to f(x) > f(y)$).

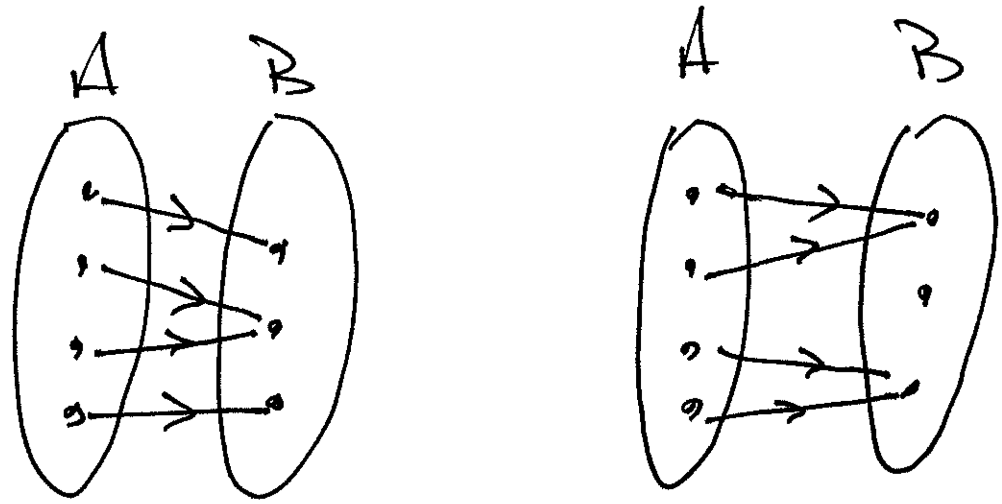
<u>Defn</u>

A function $f : A \to B$ is <u>Surjective</u> (also <u>onto</u>) iff

$$\text{range}(f) = B$$

equivalently

$$\forall y \in B \ \exists x \in A : \ y = f(x)$$



surjective   not surjective

Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ not surjective

$-1 \in \mathbb{R}$ but $-1 \notin \text{range}(f)$

so $\text{range}(f) \neq \mathbb{R}$

Ex. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is surjective

Proof: show $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R} : y = g(x)$.

Let $y \in \mathbb{R}$. define $x = \frac{y-3}{2}$.

then $g(x) = g\left(\frac{y-3}{2}\right)$

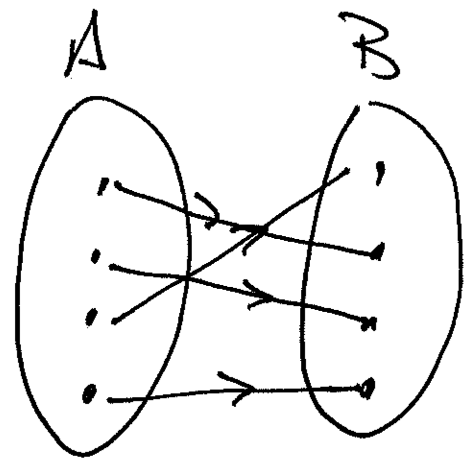
$$= 2\left(\frac{y-3}{2}\right) + 3$$

$$= (y-3) + 3$$

$$= y. \quad \blacksquare$$

Defn

A function $f : A \to B$ is called Bijective (one-to-one and onto) iff it is Both injective and surjective.



Ex. $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x+3$ is Bijective.

Defn

The composition of functions $f: A \to B$ and $g: B \to C$ is denoted: $g \circ f: A \to C$

is: $g \circ f(x) = g(f(x))$ for any $x \in A$.

Ex. $A = B = C = \mathbb{R}$

$$f(x) = x^2$$
$$g(x) = 2x + 3$$

$$g \circ f(x) = g(f(x)) = g(x^2) = 2x^2 + 3$$

$$f \circ g(x) = f(g(x)) = f(2x+3) = (2x+3)^2$$

note: even if both $f \circ g$ and $g \circ f$ are defined, in general $f \circ g \neq g \circ f$.

Theorem

let $f : A \to B$, $g : B \to C$. Then

(a) if f, g are injective, then so is $g \circ f$

(b) if f, g are surjective, then so is $g \circ f$

(c) if f, g are bijective, then so is $g \circ f$.

Proof of (a):

Let $x_1, x_2 \in A$ and Assume that $g \circ f(x_1) = g \circ f(x_2)$. must show $x_1 = x_2$.

we have: $g(f(x_1)) = g(f(x_2))$ defn. of $\circ$

$\therefore$ $f(x_1) = f(x_2)$ since g is inj.

$\therefore$ $x_1 = x$ since f is inj.

$\blacksquare$

<u>Exercise</u>: Prove (b) and (c)

<u>Exercise</u> Prove composition is associative: $f\circ(g\circ h) = (f\circ g)\circ h$.

<u>Invertibility</u>

Define for sets A, B

$$i_A : A \to A \text{ by } i_A(x) = x$$

$$i_B : B \to B \text{ by } i_B(y) = y$$

Note for any $f : A \to B$

$$f\circ i_A = f = i_B\circ f$$

called <u>identity functions</u>

<u>Defn</u>

A function $f: A \to B$ is <u>invertible</u> iff there exists $g: B \to A$ such that

$$f \circ g = i_B \quad \text{and} \quad g \circ f = i_A$$

g is called an <u>inverse function</u> for f

Theorem

If f is invertible, then its inverse g is unique.

notation: $g = f^{-1}$.

Proof.

suppose there exist $g_1 : B \to A$ and $g_2 : B \to A$ satisfying

- $f \circ g_1 = i_B$ and $g_1 \circ f = i_A$

and

- $f \circ g_2 = i_B$ and $g_2 \circ f = i_A$.

Then

$$g_1 = i_A \circ g_1$$
$$= (g_2 \circ f) \circ g_1$$
$$= g_2 \circ (f \circ g_1)$$
$$= g_2 \circ i_B$$
$$= g_2$$

$\therefore \; g_1 = g_2$ ∎

<u>Theorem</u>

$f: A \to B$ is invertible iff it is bijective

<u>Proof</u>

($\Longrightarrow$)

Assume f is invertible, so f^{-1} exists.

- <u>f is injective</u>

let $x_1, x_2 \in A$ and assume

$$f(x_1) = f(x_2)$$

$$f^{-1}(f(x_1)) = f^{-1}(f(x_2))$$

$$f^{-1} \circ f(x_1) = f^{-1} \circ f(x_2)$$

$$i_A(x_1) = i_A(x_2)$$

$$x_1 = x_2$$