Backward Reasoning

Ex. #14 p. 100

Theorem (AGM)
Arithmetic - Geometric mean inequality
Any two distinct positive $x, y \in \mathbb{R}$ satisfy

$$\underbrace{\sqrt{xy}}_{\text{G.M.}} < \underbrace{\frac{x+y}{2}}_{\text{A.M.}}$$

How to discover a proof?

$$\sqrt{xy} < \frac{x+y}{2}$$

$$xy < \frac{(x+y)^2}{4}$$

$$4xy < x^2 + 2xy + y^2$$

$$0 < x^2 - 2xy + y^2$$

$$0 < (x-y)^2$$

<u>Exercise</u> write proof in logically forward direction

2.1 Sets

Defn

A set is an unordered collection of objects, called its elements or members.

Notation write $x \in S$ to mean x is an element of set S.

Specify a set by listing members between $\{\ldots\}$

Ex.

- $\{1,2,3\} = \{2,1,3\} = \cdots = \{3,2,1\}$
- $\{1,2,3,\ldots\ldots,10\}$
- $\{1,2,3,\ldots\ldots\ldots\}$

Special sets

$$\mathbb{N} = \{0,1,2,3,\ldots\ldots\}$$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

$\mathbb{Q} = \{\text{rational numbers}\}$

$\mathbb{R} = \{\text{real numbers}\}$

$\mathbb{C} = \{\text{complex numbers}\}$

Two sets are equal iff they contain exactly the same members

$A = B$ iff $\forall x : x \in A \leftrightarrow x \in B$

Can specify a set with set-builder notation. Let $P(x)$ be a prop. function. Then

$$S = \{x \in U \mid P(x)\}$$

is the set of x in U such that $P(x)$ is true.

Also write

$$S = \{x \mid P(x)\}$$

Ex.

$$\mathbb{Q} = \{ x \in \mathbb{R} \mid \exists a, b \in \mathbb{Z} : (b \neq 0) \wedge (x = \tfrac{a}{b}) \}$$

$$\mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \}$$

Ex.

$$S = \{ n \in \mathbb{Z} \mid 0 \leq n \leq 5 \}$$

$$= \{ 0, 1, 2, 3, 4, 5 \}$$

Defn The empty set is the set with no members

$$\emptyset = \{ \}$$

<u>Defn</u>

We say set A is a <u>Subset</u> of set B iff every member of A is a member of B.

$$\forall x : x \in A \rightarrow x \in B$$

<u>Notation</u>: $A \subseteq B$

<u>Defn</u> We say A is a <u>Proper subset</u> of B iff $A \subseteq B$ and $A \neq B$. <u>Notation</u>: $A \subsetneq B$

(Book uses $A \subset B$)

observe: for any set S

- $\emptyset \subseteq S$:

  $\forall x: x \in \emptyset \to x \in S$

  note: $F \to p$ is a tautology

- $S \subseteq S$:

  $\forall x: x \in S \to x \in S$

  note: $p \to p$ is a tautology

Remark

the word 'contains' has two meanings

- $x \in S$: S contains x (as member)
- $A \subseteq S$: S contains A (as subset)

Ex. let $S = \{1, 2\}$

Subsets of S : $\phi, \{1\}, \{2\}, \{1, 2\}$

Ex. $S = \{1, 2, 3\}$

subsets: $\phi, \{1\}, \{2\}, \{3\},$

$\{1,2\}, \{1,3\}, \{2,3\},$

$\{1,2,3\}$

Defn

The set of all subsets of S is called the Power set of S.

notation: $\mathbb{P}(S)$.

Ex. $S = \{1, 2, 3\}$

$$\mathbb{P}(S) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

Defn. If $n \in \mathbb{N}$ and S has n distinct members, then we say S is finite an n is the cardinality of S.

notation: $|S| = n$

If S is not finite, it's called <u>infinite</u>.

<u>Question:</u>

- can an infinite set be a subset of a finite set?

  <u>no</u>

- can an infinite set be a member of a finite set?

  <u>Yes</u>: $\{\mathbb{Z}, \mathbb{R}, \{1,2,3\}, 8\}$

Theorem

If S is a finite set, then so is $\mathbb{P}(S)$ and

$$|\mathbb{P}(S)| = 2^{|S|}$$

Ex. $S = \{1, 2, 3\}$, $|S| = 3$

$$|\mathbb{P}(S)| = 8 = 2^3 = 2^{|S|}.$$

Defn

an ordered collection of n objects is called an ordered n-tuple.

notation: $(x_1, x_2, \ldots, x_n)$

$n=2$: (x_1, x_2) ordered pair

$n=3$: (x_1, x_2, x_3) " triple

$\vdots$

Two ordered n-tuples are equal iff they have same elements in same order.

Ex $(x_1, x_2, x_3) = (y_1, y_2, y_3)$

iff $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$.

Defn

The Cartesian Product of sets A, B is the set of all ordered pairs (x, y) with $x \in A$, $y \in B$

$$A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$$

Ex $A = \{1, 2\}$, $B = \{a, b\}$

$$A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$$

$$B \times A = \{(a,1), (b,1), (a,2), (b,2)\}$$

in general $A \times B \neq B \times A$.

Theorem

If A, B are finite sets, so is $A \times B$ and

$$|A \times B| = |A| \cdot |B|$$

More generally the Cartesian Product of set $A_1, A_2, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i,\ 1 \le i \le n\}$$

And

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Russle's Paradox (P. 146 #46)

let $\mathcal{S} = \{$all sets$\}$. Observe

$\mathcal{S} \in \mathcal{S}$ but $\emptyset \notin \emptyset$. so some

sets are members of themselves,

and some not.

Define: $R = \{A \in \mathcal{S} \mid A \notin A\}$

Either $R \in R$ or $R \notin R$. But

$$R \in R \rightarrow R \notin R$$

and

$$R \notin R \rightarrow R \in R$$

Another Paradox

Ex $A = \{B\}, \ B = \{A\}$