

CSE 107

Lab Assignment 5

In this project you will construct the CDF of a random variable $Z = Y_1 + Y_2$, where both Y_1 and Y_2 are independent exponential random variables with parameter $\lambda = 1$. You will present your CDF as a table similar in form to the Standard Normal CDF table posted on the class webpage. We will show in lecture that Z has a well-known distribution called the *Erlang* distribution. Here you will construct your CDF by performing 20,000 trials of the following simple experiment. Obtain samples Y_1 and Y_2 from the exponential distribution, then compute the sum $Z = Y_1 + Y_2$. Using these trials, compute the relative frequencies of the events $\{Z \leq z\}$ for all $z \in \{0.0, 0.1, 0.2, 0.3, \dots, 9.7, 9.8, 9.9\}$, which are 100 equally spaced points in the range 0.0 to 9.9, at distance 0.1 apart.

Your relative frequencies will constitute your estimates of $F_Z(z) = P(Z \leq z)$. The format of your table will resemble the following

Sum of Exponentials CDF:

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
3.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
4.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
5.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
6.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
7.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
8.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
9.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

where of course .0000 is replaced in each case with an appropriate relative frequency rounded to 4 decimal places. The intersection of the row labeled r and the column labeled c gives your estimate of $F_Z(r + c)$. So for instance, $F_Z(4.7)$ is found in row 4.0 and column .7.

Everything is clear except for one thing. How can you “sample” from the exponential distribution? What does that even mean? We know the PDF and CDF of the exponential random variable, but how do those functions enable you to obtain random values from that random variable? Most computer languages have built in libraries allowing you to sample from a continuous uniform distribution on the interval $[0, 1)$, i.e. pick a “random” real number x in the range $0 \leq x < 1$. Usually this function is called something like `random()` or `rand()`. Using such a function you can flip coins or roll dice with various weights, which is all we’ve done up to now, but *uniform* is not *exponential*. Some languages, like MATLAB or R, have the built-in capability to sample from non-uniform distributions, but C, C++, Java and Python do not. So how do we transform a uniform random variable into an exponential?

We seek a function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that if X is uniform on $[0, 1)$, then $Y = g(X)$ is exponential with parameter $\lambda = 1$. The function that does this is

$$g(x) = \ln\left(\frac{1}{1-x}\right).$$

In other words, if X is uniform on $[0, 1)$, then $Y = g(X)$ is exponential with parameter $\lambda = 1$. Thus we can use our built-in random number generator to obtain a number x in the range $0 \leq x < 1$, then plug it into $g(x)$ to get a random number y in the range $0 \leq y < \infty$. It is an interesting and useful exercise to

prove that the above function works as advertised, which we leave to the reader. More generally, it can be shown that the function

$$g(x) = \frac{1}{\lambda} \ln \left(\frac{1}{1-x} \right)$$

transforms a uniform random variable X on $[0,1)$ into an exponential random variable $Y = g(X)$ with parameter λ . Hint: assume that X has CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Then show that $Y = g(X)$ has CDF

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-\lambda y} & \text{if } y \geq 0 \end{cases}$$

With this piece of the puzzle in place, this project is in fact very straight forward. There is still much to do though, so please start early, and get help when you need it.