CSE 107
Homework Assignment 3

Section 2.2: Probability mass functions

1. Alice and Bob play a chess match in which the first player to win a game wins the match. In each game $P(\text{Alice wins}) = 0.5$, $P(\text{Bob wins}) = 0.4$, and $P(\text{draw}) = 0.1$, and the outcomes of each game are independent. If 10 successive games are drawn, then the match is declared to be drawn.

   (a) Determine the probability that Alice wins the match.
   (b) Let X be the duration of the match, i.e. the number of games played. Determine the PMF of X.

Section 2.4: Expectation, mean and variance

2. A class with 300 students consists of 250 undergraduate students and 50 graduate students. The probability that an undergraduate student gets an A is $1/3$, and the probability that a graduate student gets an A is $1/2$. Let X be the number of students who get an A. Determine the expectation $E[X]$.

   Hint: define random variables

   $$U_i = \begin{cases} 1 & \text{if undergraduate student } i \text{ gets an A} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \le i \le 250$$

   and

   $$G_j = \begin{cases} 1 & \text{if graduate student } j \text{ gets an A} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \le j \le 50$$

   Write X in terms of the U_i and G_j, then use linearity of expectation.

3. Let X be a random variable that takes integer values and is symmetric, that is,

   $$P(X = k) = P(X = -k)$$

   for all integers k.

   (a) Determine the expected value of $Y = \cos(X\pi)$ in terms of $p_X(k)$.
   (b) Determine the expected value of $Z = \sin(X\pi)$.

4. A particular binary data transmission and reception device is prone to some error when receiving data. Suppose that each bit is read correctly with probability p. Find the smallest value of p such that when 10,000 bits are received, the expected number of errors is at most 10.

5. Let N be a nonnegative integer-valued random variable. Show that

$$E[N] = \sum_{i=1}^{\infty} P(N \geq i)$$

6. Two coins are tossed simultaneously until their results are different, i.e. one is Heads and the other Tails. The first coin has $P(\text{Head}) = p$ and the second $P(\text{Head}) = q$. All tosses are assumed to be independent. Let X be the number of tosses. (Note: for purposes of counting tosses, a simultaneous toss of both coins counts as 1 toss.) Find the PMF, expected value and variance of X.

Section 2.5: Joint PMFs of multiple random variables

7. The UCSC football team wins any one game with probability p, and loses it with probability $1 - p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2.

8. Let X and Y be discrete random variables whose joint PMF $p_{X,Y}(x, y)$ is uniform over the set of integers x and y satisfying $1 \leq x \leq 4$ and $-1 \leq y - x \leq 1$.

   (a) Determine the marginal PMF $p_X(x)$ of X and its mean $E[X]$.
   (b) Determine the marginal PMF $p_Y(y)$ of Y and its mean $E[Y]$.
   (c) Determine the mean of $Z = X + 2Y$.

Section 2.6: Conditioning

9. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n|k) = \begin{cases} \frac{1}{k} & \text{if } n = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the joint PMF of K and N.
   (b) Find the marginal PMF of N.
   (c) Find the conditional PMF of K given that $N = 2$.
   (d) Find the conditional mean and variance of K, given that he bought at least 2 but no more than 3 books.
   (e) The cost of each book is a random variable with mean \$30. What is the expected value of his total expenditure? Hint: Condition on the events $\{N = 1\}$, $\{N = 2\}$, $\{N = 3\}$ and $\{N = 4\}$ and use the total expectation theorem.

Section 2.7: Independence

10. A fair 6-sided die is rolled until each of the outcomes $\{1, 2, 3, 4, 5, 6\}$ occurs at least once, where each roll is independent. Let X be the number of rolls performed. Determine the expected value of X. (Hint: call a roll a *success* if the number on top of the die occurs for the first time, and let X_i denote the number of rolls after the i^{th} success, up to and including the $(i+1)^{\text{st}}$ success, for $1 \leq i \leq 5$. For instance, if the sequence $(\mathbf{6}, 6, \mathbf{2}, 6, \mathbf{1}, 1, \mathbf{5}, 1, 5, 5, \mathbf{4}, 4, 4, \mathbf{3})$ is rolled, then $X_1 = 2$, $X_2 = 2$, $X_3 = 2$, $X_4 = 4$, $X_5 = 3$ and $X = 14$. Express X in terms of the the X_i, then show that each X_i is geometric and find its parameter.)