CSE 107 3-7-26

Supplemental Lecture

Poisson Process .....

Define

$$Y_K = T_1 + T_2 + \ldots + T_K$$

$$= \text{time to } K^{th} \text{ arrival} \quad (K = 1, 2, \ldots)$$

Theorem

$$f_{Y_K}(t) = \frac{\lambda^K t^{K-1} e^{-\lambda t}}{(K-1)!} \qquad (t \geq 0)$$

Erlang distribution of order K

Exercise

Prove that the sum of K indep. Exponential(λ) r.v.s is Erlang of order K.

Hint:

Use induction on K. When $K=1$ Erlang is exponential(λ). Use convolution on induction step to show

$$Y_K = \underbrace{Y_{K-1}}_{\text{ind. hyp.: is Erlang}(K-1,\lambda)} + T_K$$

Erlang(K,λ).

Summary of Bernoulli & Poisson Processes:

	Bernoulli	Poisson
time:	$t \in \{1, 2, \ldots\}$	$t \in [0, \infty)$
rate:	p/trial	λ/unit time
N_t:	Binomial(t, p)	Poisson(λt)
T_k:	Geometric(p)	Exponential(λ)
Y_k:	Pascal(p, k)	Erlang(λ, k)
X_t:	Bernoulli(p)	----------

Ex.

You receive email according to a Poisson Process of rate

$$\lambda = 2 \text{ messages/hour}$$

a.) what is Prob. of 3 new messages in next 30 minutes?

$$P(N_{\frac{1}{2}} = 3) = P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-2 \cdot \frac{1}{2}} \cdot \frac{(2 \cdot \frac{1}{2})^3}{3!}$$

$$= e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) what is Prob. of no new messages in 30 minutes?

$$P(N_{\frac{1}{2}}=0) = P_{N_{\frac{1}{2}}}(0)$$

$$= e^{-2\cdot\frac{1}{2}} \cdot \frac{(2\cdot\frac{1}{2})^0}{0!}$$

$$= e^{-1} = \boxed{.3679}$$

c.) what is expected time of your 5th new message?

$$E[Y_5] = E[T_1 + \cdots + T_5]$$

$$= E[T_1] + \cdots + E[T_5]$$

$$= \frac{1}{\lambda} + \cdots + \frac{1}{\lambda}$$

$$= 5\cdot\frac{1}{2} = \boxed{\frac{5}{2} = 2.5 \text{ hours}}$$

The sum of Poisson r.v.s

Let X, Y be independent Poisson(λ_1) and Poisson(λ_2), resp. Let

$$Z = X + Y$$

Then Z is Poisson$(\lambda_1 + \lambda_2)$.

Proof.

We have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \qquad (k = 0, 1, 2, \dots)$$

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \qquad (k = 0, 1, \dots)$$

note: $\text{range}(Z) = \{0, 1, 2, \dots\}$

Thus

$$p_Z(n) = (p_X * p_Y)(n)$$

$$= \sum_{k=0}^{\infty} p_X(k) \cdot p_Y(n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^{k}}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k!)} \lambda_1^{k} \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \lambda_1^{k} \cdot \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^{n}}{n!} \quad (n = 0, 1, \ldots)$$

$\therefore$ Z is Poisson $(\lambda_1+\lambda_2)$. $\blacksquare$

Ex. (6.2 #8)

During rush hour (8–9 am), accidents occur as a Poisson Process with rate $\lambda_1 = 5/\text{hour}$.

During Period 9–11 am, accidents occur as a Poisson Process with rate $\lambda_2 = 3/\text{hour}$.

What is the PMF of total # of accidents in 8–11 am.

Solution

Let N_1 = # accidents in 8–9

N_2 = " " " 9–11

Let

$$N = N_1 + N_2$$

Then N_1 is Poisson$(\underbrace{5\cdot 1}_{5})$

and N_2 is Poisson$(\underbrace{3\cdot 2}_{6})$.

Thus N is Poisson$(\underbrace{5+6}_{11})$, so

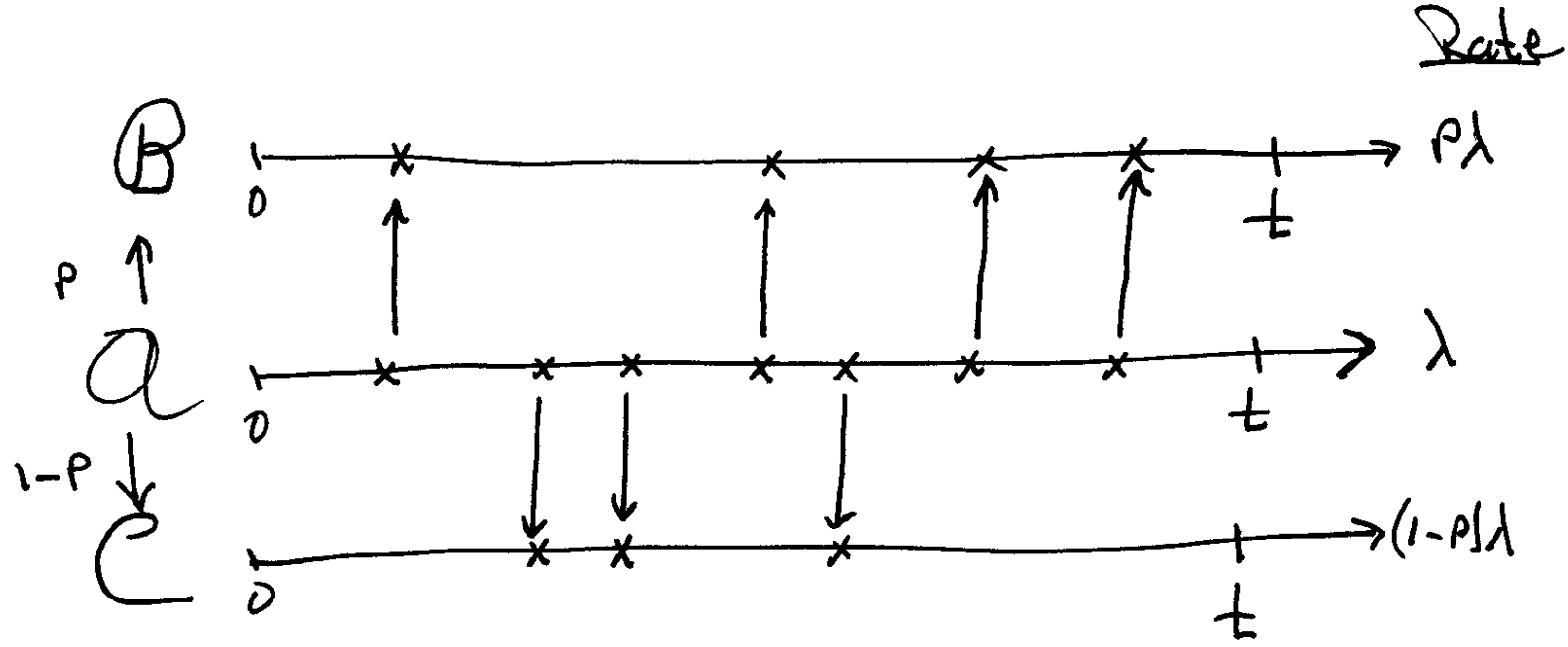
$$\boxed{P_N(n) = e^{-11}\cdot \frac{11^n}{n!}}$$

<u>splitting a Poisson Process</u>

let

$\mathcal{A}$ be a Poisson Process of rate λ

we split $\mathcal{A}$ into two Processes $\mathcal{B}$ and $\mathcal{C}$, using a coin with Prob. of head p.



What kind of Processes are B and C?

Let

N = # arrivals a in $[0,t]$

X = " " B in "

Y = " " C " "

We know N is Poisson (λt):

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

$$P_X(k) = P(X=k)$$

$$= \sum_{n=k}^{\infty} \underbrace{P(X=k \mid N=n)}_{\text{Binomial}(n,p)} \underbrace{P(N=n)}_{\text{Poisson}(\lambda t)}$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \cdot p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \cdot (\lambda t)^n}{(n-k)!} \quad \left\{ \begin{array}{c} \text{Replace} \\ n \leftarrow n+k \end{array} \right\}$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)\lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\lambda t} \cdot e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!}$$

Thus X is Poisson$(p\lambda t)$.

We see that Ⓑ satisfies (a), (b), (c) in definition of Poisson Process, so Ⓑ is a Poisson Process, rate $= p\lambda$.

Similarly C is a Poisson Process at rate $(1-p)\lambda$.

(see ex. 6.13 on p. 318).

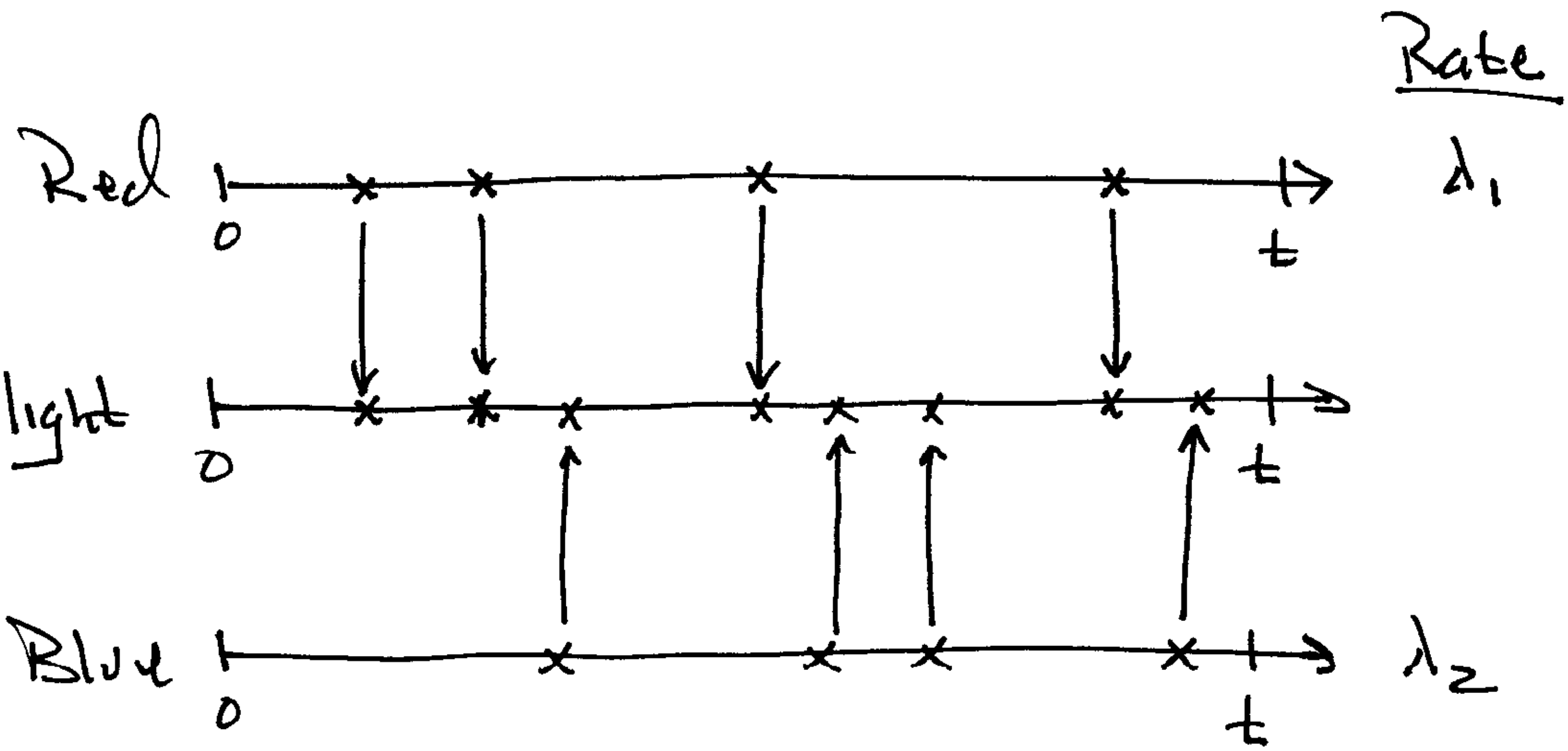
Merging Poisson Processes.

Ex. An observer sees two lights (Red and Blue) that flash at random times. Assume

$$\begin{cases} \text{Red: Poisson Process, rate} = \lambda_1 \\ \text{Blue: } \quad \text{"} \qquad \text{"} \quad \text{, rate} = \lambda_2 \end{cases}$$

The observer is colorblind, so only sees flashing lights.

What kind of Process does the observer see?



Let

$X = \#$ Red lights in $[0,t]$

$Y = \#$ Blue " " "

$M = \#$ lights " "

Know: X is Poisson$(\lambda_1 t)$

Y " Poisson$(\lambda_2 t)$

Thus $M = X + Y$, so M is Poisson$(\underbrace{\lambda_1 t + \lambda_2 t}_{(\lambda_1+\lambda_2)t})$.

Thus light Process is a Poisson Process of rate $= \lambda_1 + \lambda_2$.

Ex

suppose observer sees a single flash. Whats the Prob. that it was Red?

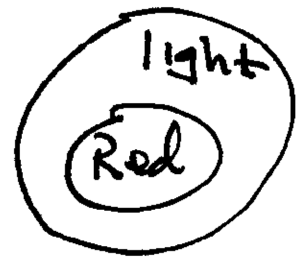
Solution 1

use small interval Probabilities.

Let $\delta > 0$ be the width of a small time interval containing the flash. We seek

$$P(1 \text{ Red} \mid 1 \text{ light})$$

$$= \frac{P(1 \text{ Red}, 1 \text{ light})}{P(1 \text{ light})}$$



$$= \frac{P(1 \text{ Red})}{P(1 \text{ light})}$$

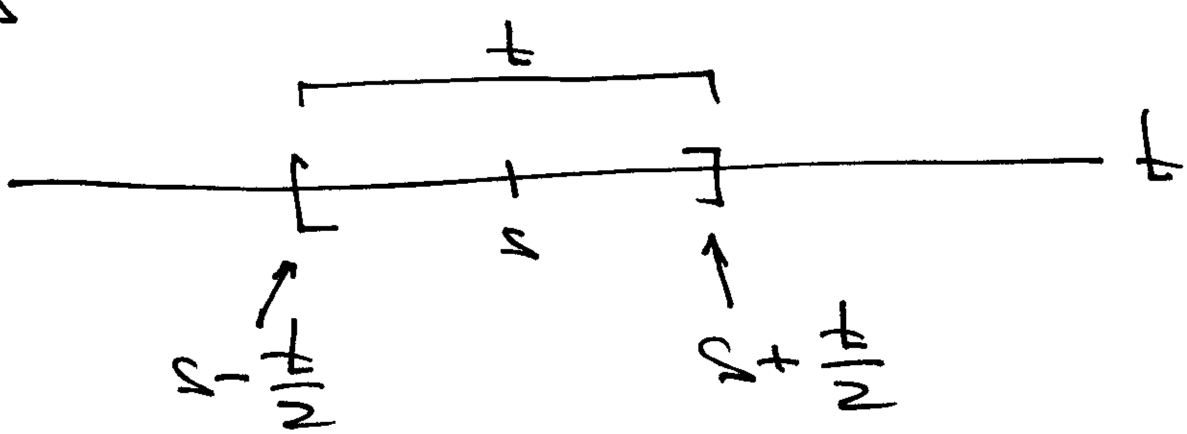
$$= \frac{\lambda_1 \cancel{\delta}}{(\lambda_1 + \lambda_2) \cancel{\delta}}$$

$$= \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

Solution 2

not using small interval Prob.

Suppose light flashes at time s. Consider an interval of length t centered at s.



i.e $I_t = \left[s - \frac{t}{2}, s + \frac{t}{2}\right]$

Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t, 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{(\lambda_1+\lambda_2)t}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[t \to 0]{\text{as}} \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$