CSE 107 3-12-26

- Final Exam

  Wed. March 18

  8:00 – 10:00 am

  Come early!

7.2 Classification of states

Defn

We say that state j is *accessible* from state i (or that j is *reachable* from i) iff

$$r_{ij}(n) > 0 \quad \text{for some } n \geq 1$$

Equivalently, there exists a sequence of states $i, i_1, i_2, \ldots, i_{n-1}, j$ such that the transitions

$$(i) \to (i_1) \to (i_2) \to \cdots \to (i_{n-1}) \to (j)$$

all have positive probability.

notation: $(i) \dashrightarrow (j)$

Defn

Let $A(i)$ denote the set of states accessible from i, i.e.

$$A(i) = \{\, j \in S \mid (i) \dashrightarrow (j) \,\}$$

<u>Defn</u>

A state $i \in S$ is called <u>recurrent</u> if for every j accessible from i, we also have i accessible from j.

Equivalently

$$j \in A(i) \quad \text{iff} \quad i \in A(j)$$

i.e.

$$i \dashrightarrow j \quad \text{iff} \quad j \dashrightarrow i$$

Thus if we depart a recurrent state, there is a positive probability that we return, and so after a long enough time, we will return.

Hence a recurrent state, if it is visited at all, <u>will be visited infinitely often</u>.

<u>Defn</u>

A state i is called <u>transient</u> iff it is not recurrent. Thus i is transient iff there exists a state j with

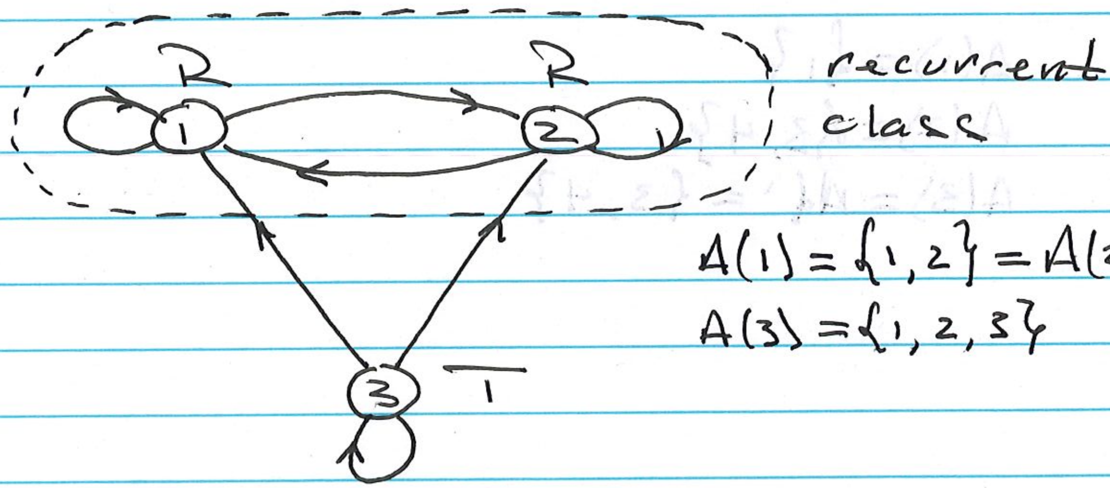
$$j \in A(i) \quad \text{and} \quad i \notin A(j)$$

Thus after a visit to a transient state i, there is a positive probability of entering such a state j. Given enough time, this is certain to happen.

Hence a transient state will only be visited a finite number of times.

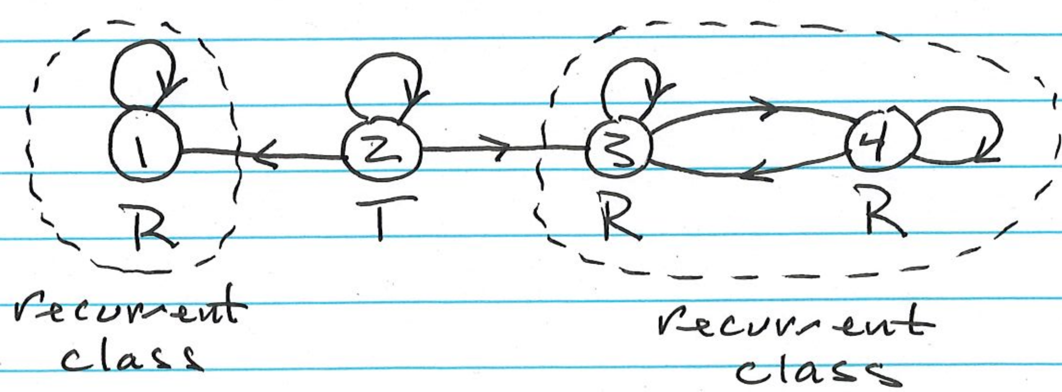
<u>Note</u>: transience and recurrence are determined by the arcs in a state transition diagram, not by the actual probabilities P_{ij} on each arc.

Ex.



$A(1) = \{1, 2\} = A(2)$
$A(3) = \{1, 2, 3\}$

Ex.



Note that for any states i, j we have

$$j \in A(i) \Rightarrow A(j) \subseteq A(i)$$

and

$$i \in A(j) \Rightarrow A(i) \subseteq A(j)$$

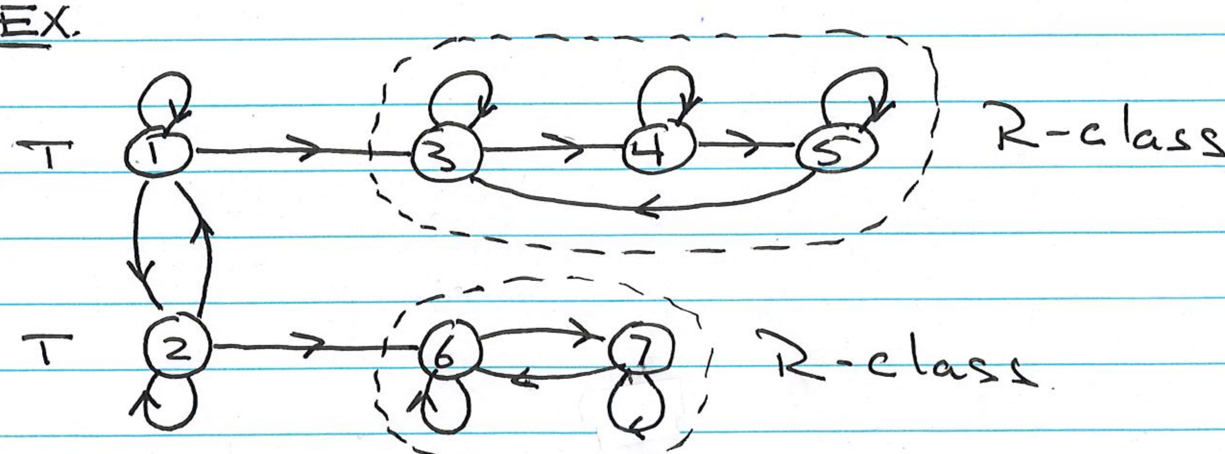
Therefore if i is recurrent, then

$$\left.\begin{array}{c} j \in A(i) \Rightarrow A(j) \subseteq A(i) \\ \Updownarrow \\ i \in A(j) \Rightarrow A(i) \subseteq A(j) \end{array}\right\} \Rightarrow A(i) = A(j)$$

<u>Defn</u>

If i is a recurrent state, the set $A(i) \subseteq S$ is called a <u>recurrent class</u>.

<u>EX.</u>



Problem
Exercise (see #6 on p. 381)
Let i be a transient state. Prove there exists a recurrent state j such that

$$T \ (i) \dashrightarrow (j) \ R$$

i.e.

$$j \in A(i)$$

Theorem: Markov Chain Decomposition

- A Markov chain can be decomposed into one or more recurrent classes, and possibly some transient states.

- A recurrent state is accessible from all states in its class, but not from recurrent states in other classes.

- A transient state is not accessible from any recurrent state

- At least one recurrent state is accessible from a given transient state.

Theorem

(a) When a Markov chain enters a recurrent class, it stays within that class, and all states in the class will be visited infinitely often

(b) If the initial state of a Markov chain is transient, then its trajectory consists of a finite sequence of transient states, followed by an infinite sequence of recurrent states belonging to the same class.

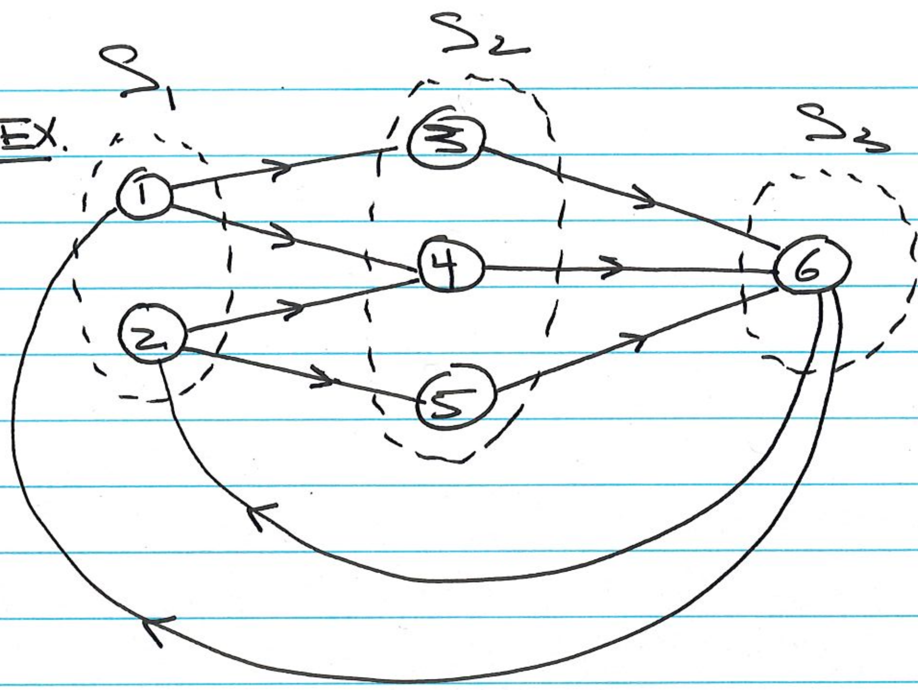
Defn

A recurrent class is called *Periodic* if its states can be grouped into $d \geq 2$ disjoint subsets $S_1, \ldots, S_d$ such that all transitions from S_k lead to S_{k+1} $(1 \leq k < d)$ and S_d leads to S_1.

$$\begin{tikzcd} S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \cdots \arrow[r] & S_{d-1} \arrow[r] & S_d \arrow[lllll, bend left] \end{tikzcd}$$

We call d the *period* of the class

Ex.



Period = 3

observe that in this example, if $X_0 \in S_1$, then

$$X_n \in S_1 \quad \text{iff} \quad n \equiv 0 \pmod{3}$$
$$X_n \in S_2 \quad \text{iff} \quad n \equiv 1 \pmod{3}$$
$$X_n \in S_3 \quad \text{iff} \quad n \equiv 2 \pmod{3}$$

Defn

A recurrent class is called aperiodic iff it is not periodic. Equivalently, there exists a time $n \geq 0$ such that

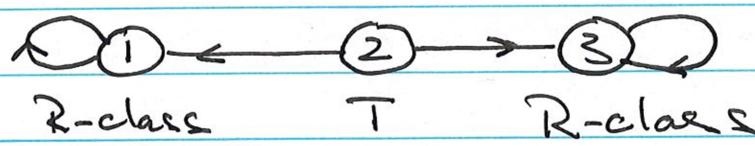
$$r_{ij}(n) > 0 \quad \text{for all} \quad i,j \in R$$

Ex. add a self loop to any state in the previous example.

7.3 steady-state behavior

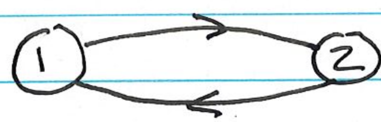
we do not expect the n-step transition probabilities to have limits (not depending on initial state) when there are either

- multiple recurrent classes



or

- periodic recurrent classes



we will write

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad (\text{for all } i)$$

when the limit exists, and we call π_j the <u>steady-state probability</u> of j.

In the case where these limits exist (independent of i) we interpret

$$\pi_j \approx P(X_n = j) \quad \text{(for } n \text{ large)}$$

when do the π_j exist?

<u>Theorem</u> (steady-state convergence)
Consider a Markov chain with a single recurrent class, which is aperiodic. Then for each $j \in S$:

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad \text{(for all } i\text{)}$$

exists. Furthermore, π_j are solutions to the linear system

$$(1) \qquad \sum_{k=1}^{m} \pi_k p_{kj} = \pi_j \qquad (j = 1, 2, \ldots, m)$$

$$(2) \qquad \sum_{k=1}^{m} \pi_k = 1$$

Also $\pi_j = 0$ if j is transient, and $\pi_j > 0$ if j is recurrent.

Equations (1) are called the <u>balance equations</u>, and follow directly from the Chapman-Kolmogorov equations

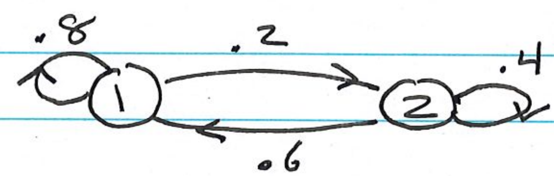
$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$

By taking limits $r_{ij}(n) \to \pi_j$ and $r_{ik}(n-1) \to \pi_k$ as $n \to \infty$.

Equation (2) is called the <u>normalization equation</u>, and follow from the fact that

$$\sum_{j=1}^{m} P(X_n = j) = 1 \quad (\text{any } n \geq 0)$$

<u>Ex</u>



$p_{11} = .8$, $p_{12} = .2$

$p_{21} = .6$, $p_{22} = .4$

$$(1) \begin{cases} \pi_1 p_{11} + \pi_2 p_{21} = \pi_1 \\ \pi_1 p_{12} + \pi_2 p_{22} = \pi_2 \end{cases}$$

$$(2) \quad \pi_1 + \pi_2 = 1$$

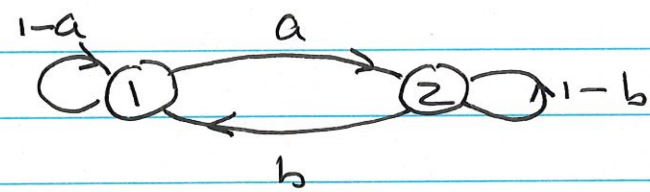
$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 = \pi_1 \iff (.6)\pi_2 = (.2)\pi_1 \\ (.2)\pi_1 + (.4)\pi_2 = \pi_2 \iff (.2)\pi_1 = (.6)\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

The 1st two give $\pi_1 = 3\pi_2$. The 3rd yields

$$3\pi_2 + \pi_2 = 1 \to 4\pi_2 = 1 \to \boxed{\pi_2 = \frac{1}{4}}$$

hence $\boxed{\pi_1 = \frac{3}{4}}$

Ex.



$$\pi_1(1-a) + \pi_2 b = \pi_1$$
$$\pi_1 a + \pi_2(1-b) = \pi_2$$

$$\pi_2 b = \pi_1 a$$

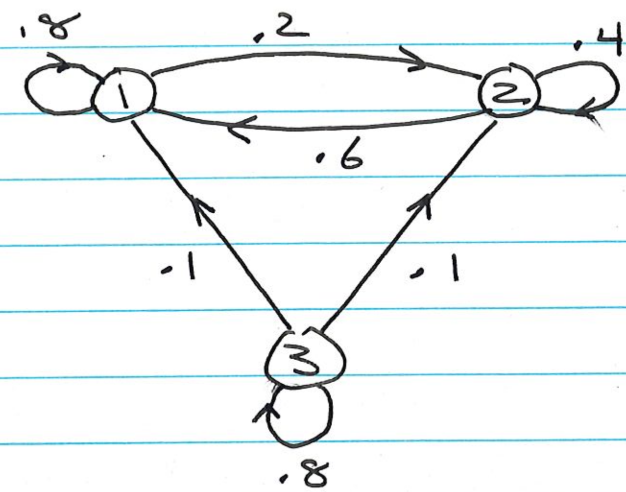
$$\therefore \quad \pi_1 = \frac{b}{a}\pi_2$$

$$\therefore \quad \frac{b}{a}\pi_2 + \pi_2 = 1$$

$$\therefore \quad \pi_2 = \frac{1}{\frac{b}{a}+1} \to \boxed{\pi_2 = \frac{a}{a+b}}$$

$$\therefore \quad \boxed{\pi_1 = \frac{b}{a+b}}$$

Ex.



$$\begin{cases} \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} = \pi_2 \\ \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} = \pi_3 \end{cases}$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 + (.1)\pi_3 = \pi_1 \\ (.2)\pi_1 + (.4)\pi_2 + (.1)\pi_3 = \pi_2 \end{cases}$$

$$\begin{cases} -2\pi_1 + 6\pi_2 + \pi_3 = 0 \\ 2\pi_1 - 6\pi_2 + \pi_3 = 0 \end{cases} \rightarrow 2\pi_3 = 0 \rightarrow \boxed{\pi_3 = 0}$$

$$\rightarrow \pi_1 = 3\pi_2$$

$$3\pi_2 + \pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}} \quad \therefore \boxed{\pi_1 = \frac{3}{4}}$$

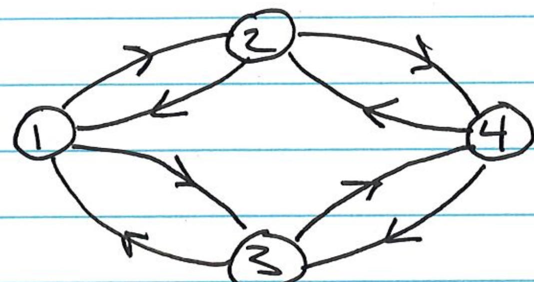
See examples 7.6 and 7.7 on Pages 355 and 356, resp.

we should have anticipated this since we know state 3 is transient

Periodicity

Ex.

a). show that the Markov chain model with transition diagram

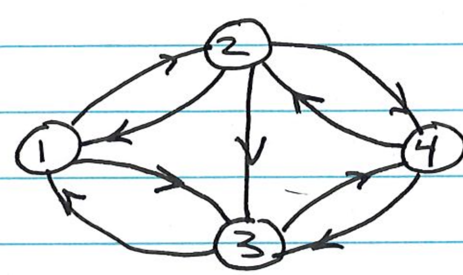


is Periodic, and find the period.

Proof

Let $S_0 = \{1, 4\}$ and $S_1 = \{2, 3\}$. Then all transitions go from S_0 to S_1, or S_1 to S_0. Thus the chain is Periodic of Period 2. $\blacksquare$

b.) Add a transition ② → ③ in the above markov chain model.



show that the resulting chain is not Periodic.

Proof

Assume the above chain is Periodic and let S_1 be the set containing state ①. Since the chain has transitions from ① to ② and from ① to ③, states ② and ③ must be in the same set. But the transition ②→③ implies that ② and ③ cannot belong to the same set. This contradiction shows that our assumption was false, and hence the above Markov chain is NOT Periodic ∎

e.) Show that the chain in (b) has steady-state probabilities.

Proof.

Observe that $A(1)=A(2)=A(3)=A(4) = \{1,2,3,4\}$, so that the chain contains a single recurrent class. By the steady-state convergence theorem, the limits

$$\pi_j = \lim_{n\to\infty} P_{ij}(n) \quad \text{(for all } i\text{)}$$

exist for $j=1,2,3,4$. ∎

d.) Determine $P(X_{1003}=4 \mid X_{1000}=1)$ for the chain in (b).

<u>solution</u>

only one path in the Markov chain goes from 1 to 4 in 3 transitions, namely $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. Thus

$$P(X_{1003}=4 \mid X_{1000}=1) = r_{14}(3) = \boxed{p_{12}\, p_{23}\, p_{34}}.$$