CSE 107 3-10-26

7.1 Discrete Markov Chains

Let S be a finite set of states

$$S = \{1, 2, 3, \ldots, m\}$$

Let $X_0, X_1, \ldots, X_n, \ldots$ be a seq. of r.v.s taking values in S.

Consider

X_n = state of Process at time n.

We Define <u>Transition Probabilities</u>

$$p_{ij} = P(X_{n+1} = j \mid X_n = i) \quad \begin{cases} \text{any } i,j \in S \\ \text{and any} \\ n \geq 0 \end{cases}$$

<u>note</u>:

- RHS does not depend on $n \geq 0$

<u>Markov Property</u>

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

$$= P(X_{n+1} = j \mid X_n = i)$$

$$= p_{ij}$$

for all $i,j \in S$, $n \geq 0$.

i.e. The memory of a Markov Process goes back only one step in time.

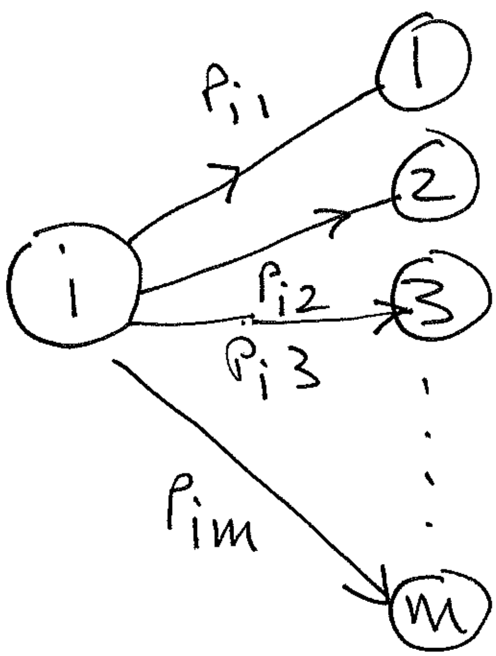
- State space S
- Transition Probabilities
- Markov Property

Constitute a Markov Chain Model.

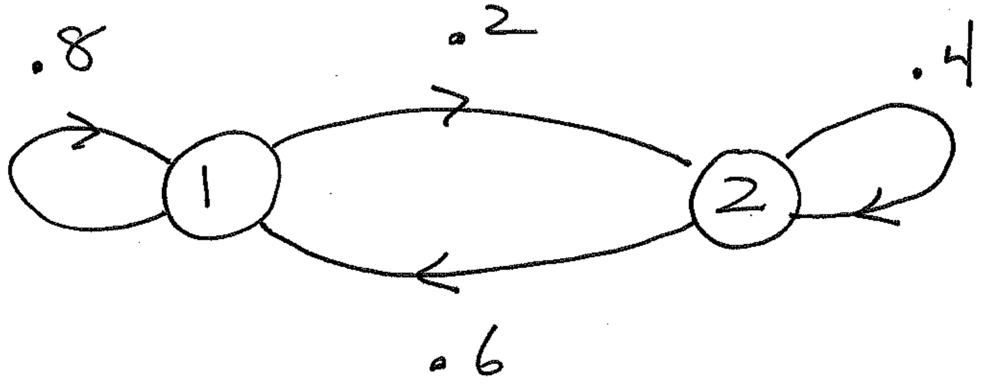
Note

- necessarily

$$\sum_{j=1}^{m} p_{ij} = 1 \quad \text{for all } i \in S.$$



Ex. $m=2$, $S=\{1,2\}$



i.e. $p_{11}=.2$, $p_{12}=.2$, $p_{21}=.6$, $p_{22}=.4$

Suppose $\boxed{X_0=1}$. Find $P(X_2=2)$

By total Prob. Thm.:

$$P(X_2=2)=P(X_2=2|X_1=1)\cdot P(X_1=1)$$

$$+P(X_2=2|X_1=2)\cdot P(X_1=2)$$

$$=p_{12}\cdot P(X_1=1)+p_{22}\cdot P(X_1=2)$$

__Note:__

$$P(X_1=1) = P(X_1=1 \mid X_0=1)\cdot\overbrace{P(X_0=1)}^{1}$$

$$+\,P(X_1=1 \mid X_0=2)\cdot\overbrace{P(X_0=2)}^{0}$$

$$= p_{11}\cdot 1 = p_{11}$$

also

$$P(X_1=2) = P(X_1=2 \mid X_0=1)\cdot\overbrace{P(X_0=1)}^{1}$$

$$+\,P(X_1=2 \mid X_0=2)\cdot\overbrace{P(X_0=2)}^{0}$$

$$= p_{12}$$

Thus

$$P(X_2=2) = p_{12}\cdot p_{11} + p_{22}\cdot p_{12}$$

$$= p_{11}\cdot p_{12} + p_{12}\cdot p_{22}$$

$$= (.8)(.2) + (.2)(.4) = \boxed{.24}$$

Similarly

$$P(X_2=1) = \cdots = p_{11}p_{11} + p_{12}p_{21}$$

$$= (.8)^2 + (.2)(.6)$$

$$= \boxed{.76}$$

In a similar manner, if $\boxed{X_0 = 2}$,

then

$$P(X_2 = 1) = P_{21}P_{11} + P_{22}P_{21} = \boxed{.72}$$

and

$$P(X_2 = 2) = P_{21}P_{12} + P_{22}P_{22} = \boxed{.28}$$

We can summarize all calculations be the Transition Probability Matrix.

$$R = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

observe that

$$R^2 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}p_{11} + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{21}p_{12} + p_{22}p_{22} \end{pmatrix}$$

$$= \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

Thus, the ij^{th} entry in R^2 is the Probability of going from state i to state j in 2 steps

Defn

A Probability Matrix is a square (i.e. $n \times n$) matrix in which each row sums to 1.

i.e. if $A = (a_{ij})$ $(1 \le i \le n, 1 \le j \le n)$, then

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for all } 1 \le i \le n$$

Exercise

Prove that the product of two probability matrices is another probability matrix.

Defn

The <u>n-step transition Probabilities</u>

are

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

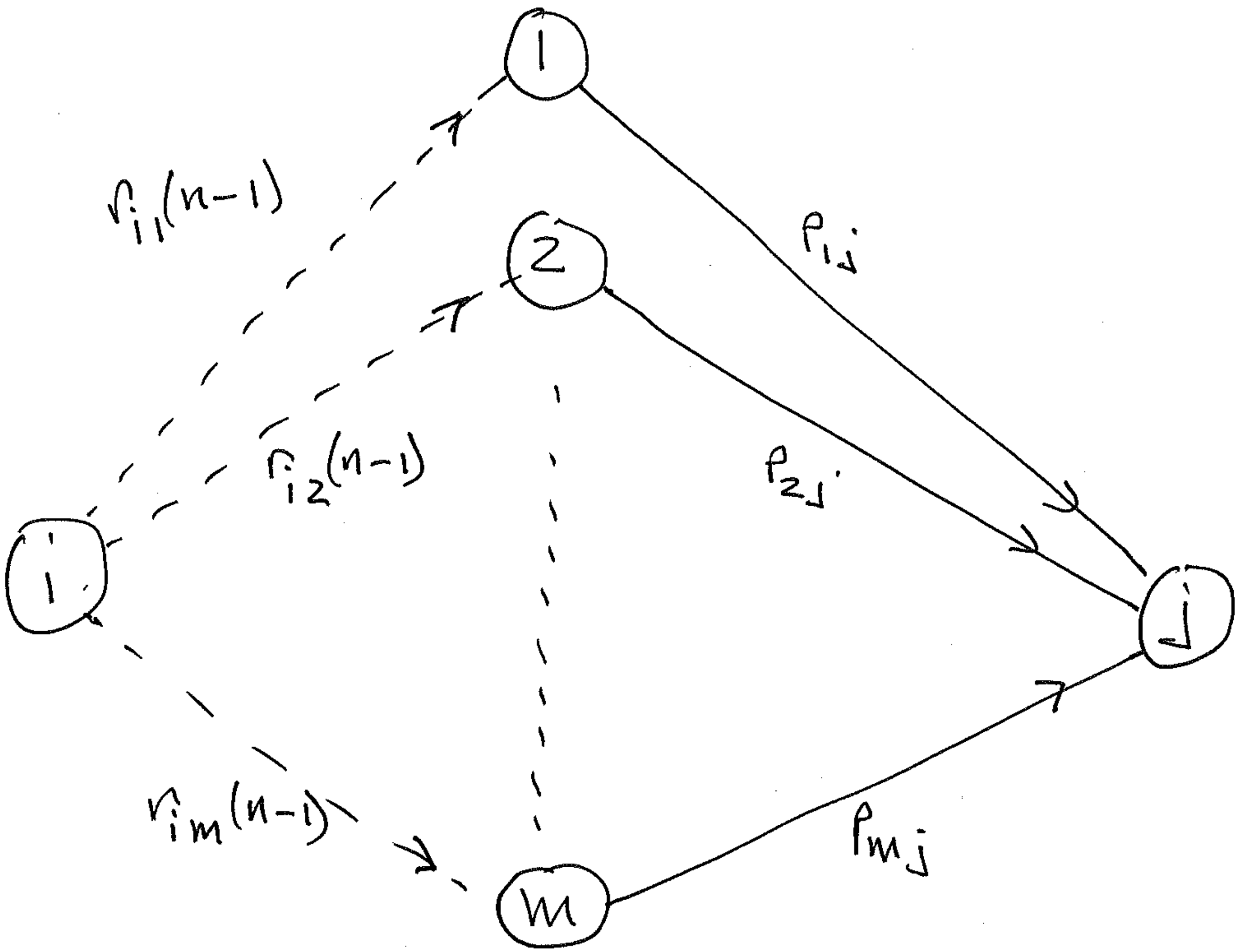
<u>note</u>: For any $k \geq 0$: $r_{ij}(n) = P(X_{n+k} = j \mid X_k = i)$

<u>note</u>: $r_{ij}(1) = p_{ij}$

Let $n \geq 2$. Then we have

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj} \quad \begin{cases} \text{all } i,j \in S \\ \text{all } n \geq 2 \end{cases}}$$

<u>Chapman-Kolmagorov equations</u>



Matrix Version of C-K Eqns

$$\boxed{\left(r_{ij}(n)\right)_{(i,j\in S)} = R^{n-1}\cdot R = R^{n}}$$

Ex. Previous

$$R = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

$$R^2 = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

$$R^3 = \cdots\cdots = \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}$$

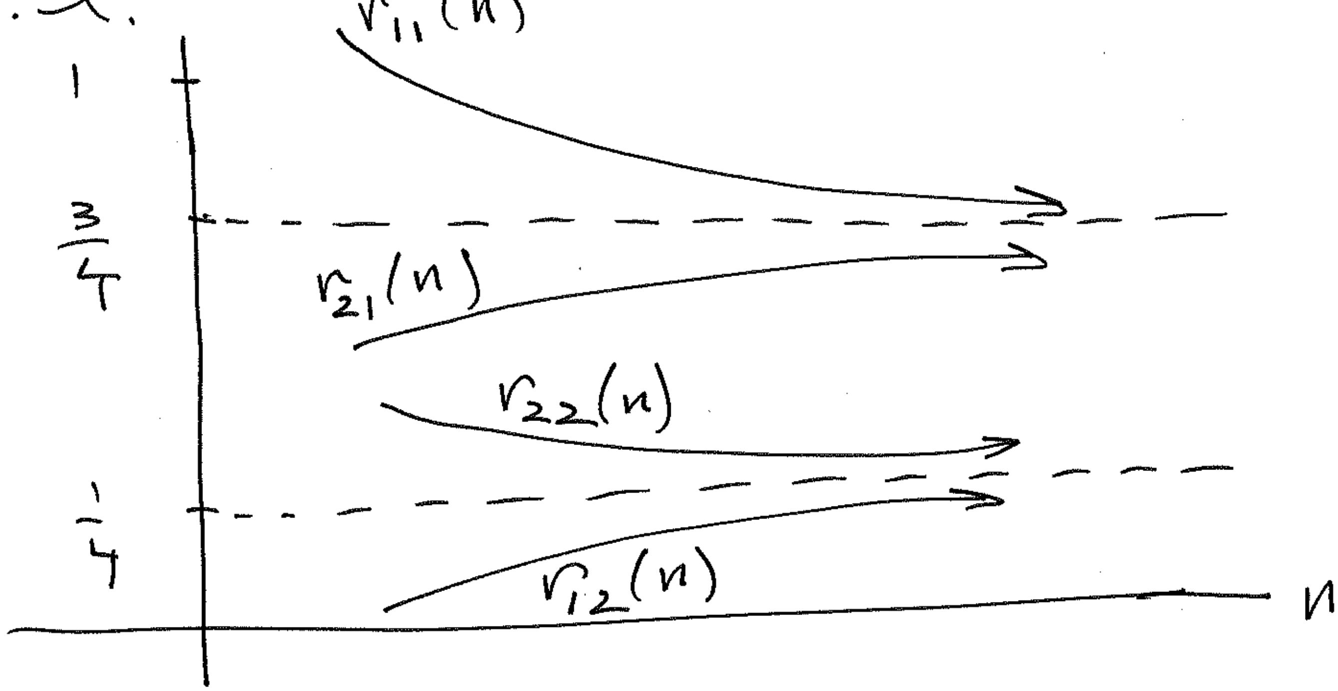
$$\vdots$$

$$R^{10} = \begin{pmatrix} .7500000256 & .2499999744 \\ .7499999232 & .2500000768 \end{pmatrix}$$

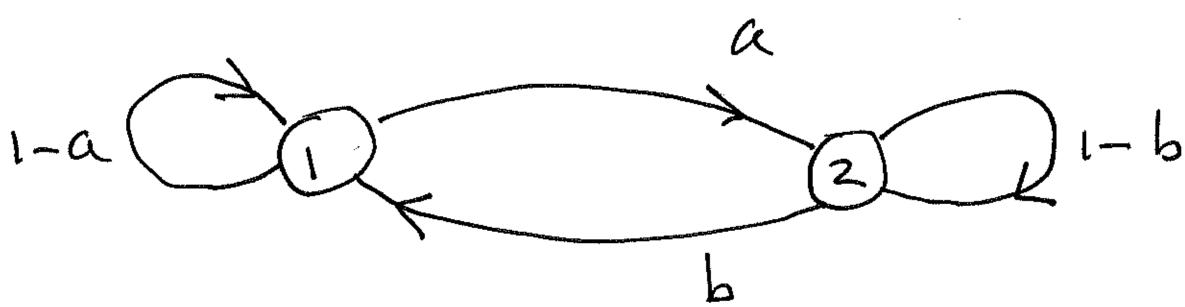
In fact

$$\lim_{n \to \infty} R^n = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix}$$

i.e.



Ex.



Here $a, b \in (0,1)$, so

$$R = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

one can show

$$\lim_{n \to \infty} P^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

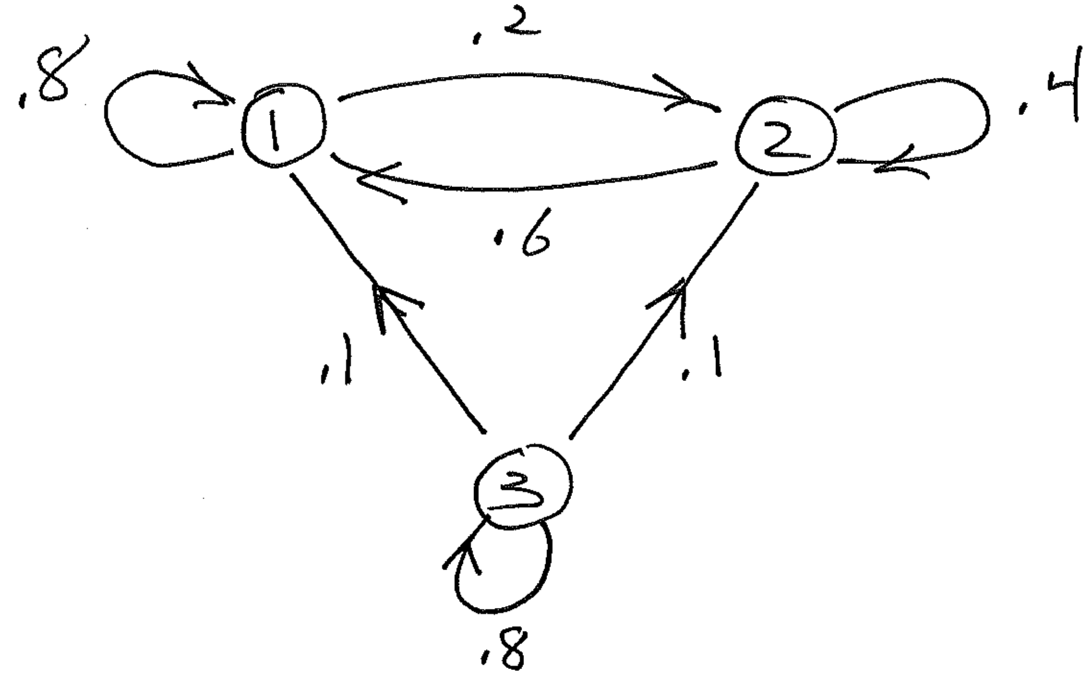
Hence

$$\lim_{n \to \infty} P(X_n = 1) = \frac{b}{a+1}$$

$$\lim_{n \to \infty} P(X_n = 2) = \frac{a}{a+b}$$

regardless of X_0.

Ex. $m = 3$, $S = \{1, 2, 3\}$



Thus

$$R = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .4 & 0 \\ .1 & .1 & .8 \end{pmatrix}$$

One can show

$$\lim_{n\to\infty} R^n = \begin{pmatrix} .75 & .25 & 0 \\ .75 & .25 & 0 \\ .75 & .25 & 0 \end{pmatrix}$$

Hence

$$P(X_n = 1) \longrightarrow \frac{3}{4}$$

$$P(X_n = 2) \longrightarrow \frac{1}{4}$$

$$P(X_n = 3) \longrightarrow 0$$

as $n \to \infty$

Regardless of initial state.

7.2 Classification of states

Defn

state j is accessible from state i iff

$$r_{ij}(n) > 0 \quad \text{for some } n \geq 1$$