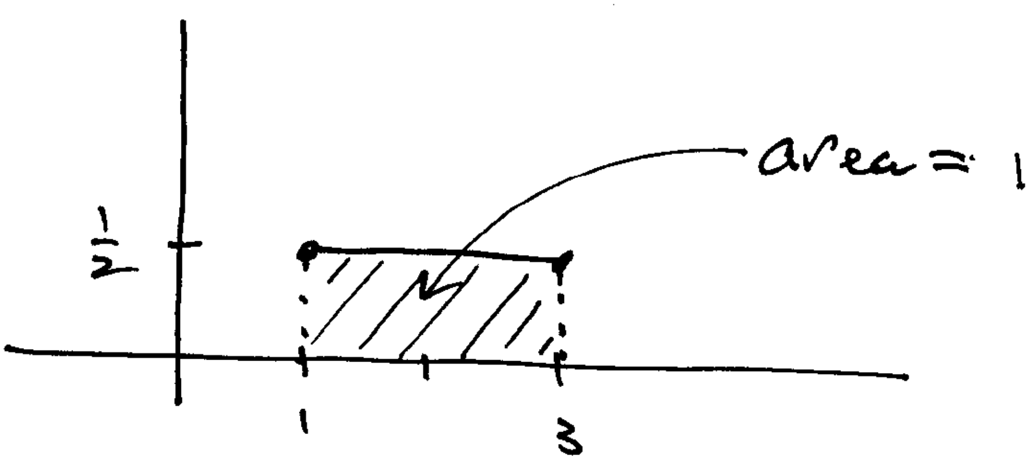
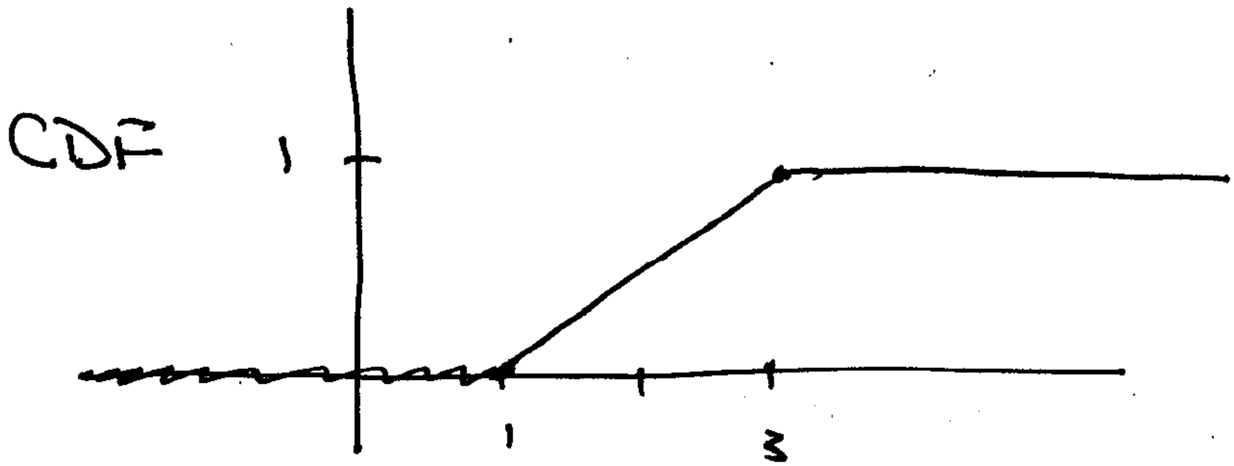
CSE 107 2-6-26

Supplemental Lecture

<u>Ex</u> Continuous uniform r.v. on $[1, 3]$

PDF : $f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$





Let $x \in [1,3]$. Then

$$F_X(x) = \int_1^x \frac{1}{2}\,dt = \frac{1}{2}t\Big|_1^x = \frac{x-1}{2}$$

Thus

$$F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x-1}{2} & \text{if } 1 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

- note: The CDF

$$F_X(x) = P(X \le x)$$

is always monotone non-decreasing, i.e.

$$x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$$

also $0 \le F_X(x) \le 1$

- if X is discrete, then $F_X(x)$ is piecewise constant

$$\boxed{F_X(k) = \sum_{i \le k} P_X(i)}$$

assume range$(X) \subseteq \mathbb{Z}$

also

$$p_X(k) = P(X = k)$$

$$= P(X \le k) - P(X \le k-1)$$

$$\boxed{p_X(k) = F_X(k) - F_X(k-1)}$$

- if X is continuous, then $F_X(x)$ is a continuous function

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt}$$

also

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

Ex.

you can take a test 3 times, and your final score X is the maximum of the individual scores: X_1, X_2, X_3.

$$X = \max(X_1, X_2, X_3)$$

Assume X_1, X_2, X_3 are independent.

Your score on test i $(i = 1, 2, 3)$ is uniform over $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, i.e.

$$P_{X_i}(k) = \begin{cases} \frac{1}{10} & \text{if } k = 1, 2, 3, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

What is the PMF $P_X(k)$ of X?

Solution

First find CDF $F_X(k)$, then difference to get PMF

$$P_X(k) = F_X(k) - F_X(k-1)$$

so

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

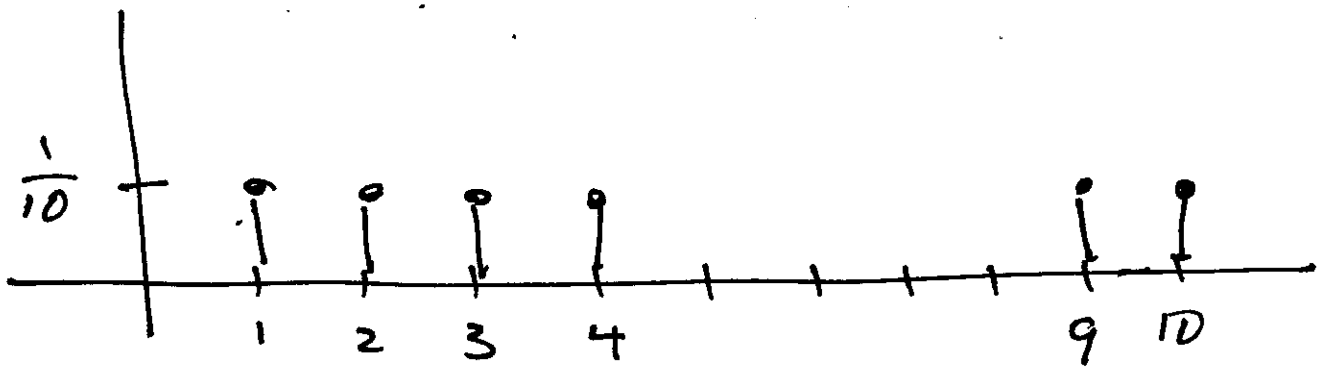
$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$$

by indep.

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

Recall for $i = 1, 2, 3$

$$P_{X_i}(k) = \begin{cases} 1/10 & k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$



so

$$F_{X_i}(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{\lfloor x \rfloor}{10} & \text{if } 1 \le x \le 10 \\ 1 & \text{if } x > 10 \end{cases}$$

Thus

$$F_X(k) = F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{k}{10}\right)^3$$

Therefore $P_X(k) = F_X(k) - F_X(k-1)$

and

$$P_X(k) = \begin{cases} \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 & \text{if } k = 1, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

$\blacksquare$

Goal: Compare Geometric(p) and Exponential(λ)

Recall if X is Geometric(p), then

$$p_X(k) = p(1-p)^{k-1} \quad (k=1,2,3,\ldots)$$

so

$$F_X(n) = P(X \le n)$$

$$= \sum_{k=1}^{n} p(1-p)^{k-1}$$

$$= \cancel{p}\,\frac{1-(1-p)^n}{\underbrace{1-(1-p)}_{\cancel{p}}}$$

$$= 1-(1-p)^n$$

<u>Recall:</u>

$$S = \sum_{k=1}^{n} a \cdot r^{k-1} = a\left(\frac{1-r^n}{1-r}\right)$$

<u>Proof</u>

$$S = a\left(\underbrace{r^0}_{1} + \cancel{r^1} + \cancel{r^2} + \cdots + \cancel{r^{n-1}}\right)$$

$$rS = a\left(\cancel{r^1} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-1}} + r^n\right)$$

$$\therefore\ S - rS = a(1-r^n)$$

$$\therefore\ (1-r)S = a(1-r^n)$$

$$S = a\,\frac{1-r^n}{1-r} \quad (\text{if } r \neq 1)$$

$\blacksquare$

If Y is Exponential(λ), then

$$f_Y(x) = \lambda e^{-\lambda x} \quad (\text{if } x \geq 0)$$

Thus for $x \geq 0$, we have

$$F_Y(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x$$

$$= -\left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x}$$

To compare F_{geo} and F_{exp}

Pick $\delta > 0$ (width of discrete time interval). Require

$$e^{-\lambda\delta} = 1 - p \qquad (\text{i.e. } p = 1 - e^{-\lambda\delta})$$

$$\therefore \quad -\lambda\delta = \ln(1-p)$$

$$\therefore \quad \delta = \frac{-\ln(1-p)}{\lambda}$$

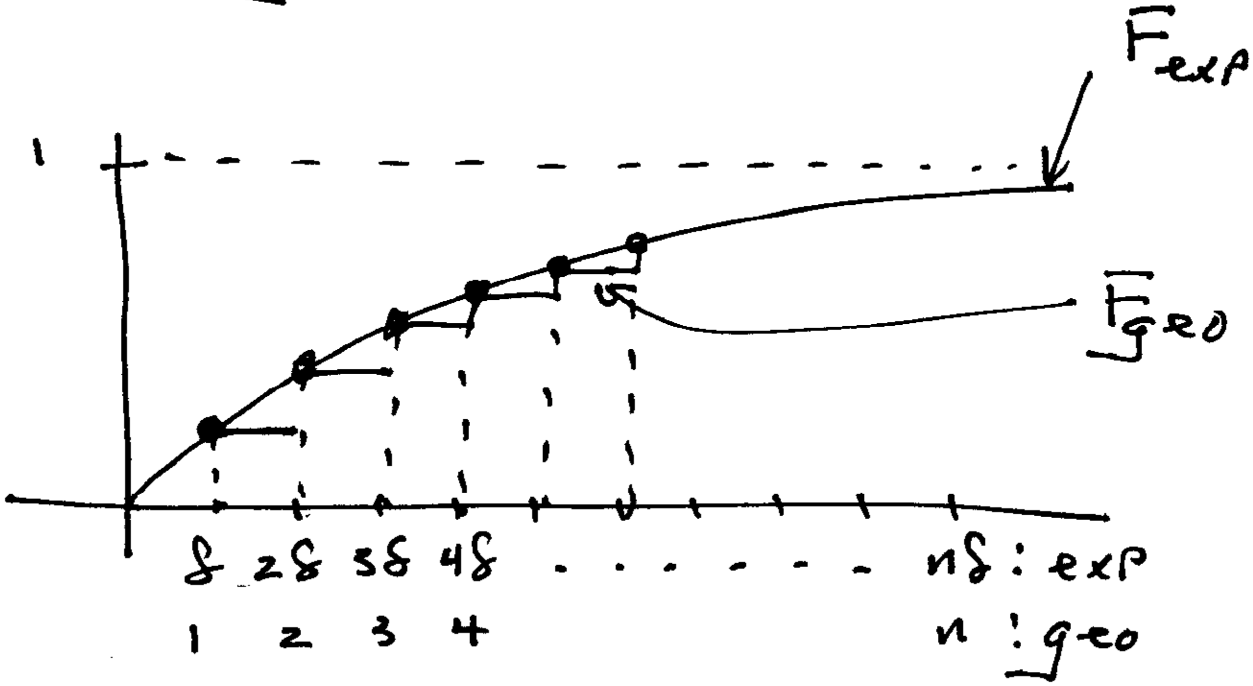
For this δ we have

$$\left(e^{-\lambda\delta}\right)^n = (1-p)^n$$

$$\therefore \quad 1 - e^{-\lambda\delta n} = 1 - (1-p)^n$$

$$\therefore \quad F_{exp}(\underbrace{\delta n}_{\substack{\uparrow \\ \text{take place} \\ \text{at } x}}) = F_{geo}(n)$$

Graphing



so if we toss a coin quickly (every δ second) with Prob.

$$P = 1 - e^{-\lambda\delta}$$

Then time of 1^{st} heads is approx exponential with λ parameter.

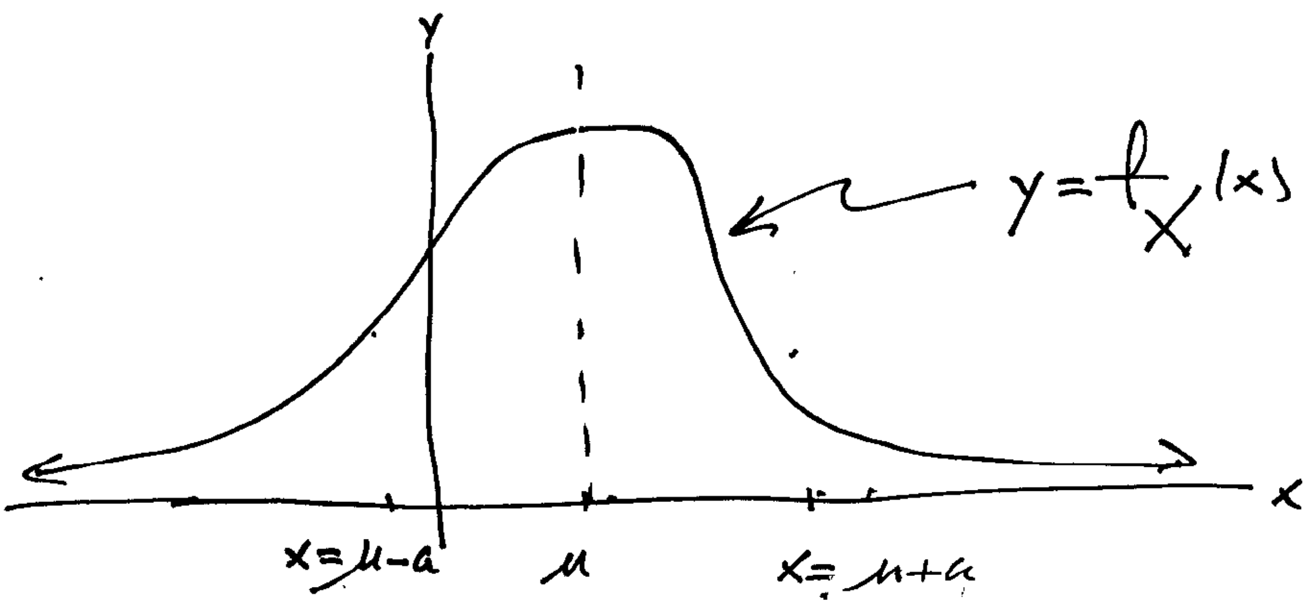
3.3 Normal Random Variables

Let X be a continuous r.v. with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad (x \in \mathbb{R})$$

we call this Normal (μ, σ)

mean std. dev.



can show:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 .$$

see handout "Gaussian Integral".

we can also show

- $E[X] = \mu$ ← follows from symmetry.
- $Var(X) = \sigma^2$ ← handout
- $\sqrt{Var(X)} = \sigma$ ← follows