CSE 107 2-5-26

<u>Ex</u> if $Y = aX + b$, then

- $E[Y] = a\,E[X] + b$

and

- $Var(Y) = a^2\,Var(X)$

<u>Exercise</u>: Prove these.

Ex. continuous Unif. on $[a, b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx = \left(\frac{1}{b-a}\right) \cdot \frac{x^2}{2} \Big|_a^b$$

$$= \frac{1}{2}\left(\frac{1}{b-a}\right)(b^2 - a^2) = \frac{1}{2} \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}}$$

$$= \boxed{\frac{a+b}{2}}$$

$$E[X^2] = \int_a^b x^2 \cdot \left(\frac{1}{b-a}\right) dx$$

$$= \underset{\text{(exercise)}}{------} = \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= ---- = \boxed{\frac{(b-a)^2}{12}}$$

Exercise

Alice's driving time.

$$f_X(x) = \begin{cases} 1/6 & \text{if } 40 \le x \le 44 \\ 1/18 & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

Find answers

$E[X] = \frac{131}{3}$

$E[X^2] = \frac{17{,}228}{9}$

$Var(X) = \frac{67}{9}$

Exponential Random Variable

Parameter : $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

check:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\lambda}\left(-\frac{1}{\cancel{\lambda}}\right)\cdot e^{-\lambda x}\Big|_0^{\infty}$$

$$= -(0-1) = \boxed{1}$$

Exercise

- $E[X] = \frac{1}{\lambda}$ ✓
- $E[X^2] = \frac{2}{\lambda^2}$
- $Var(X) = \frac{1}{\lambda^2}$

observe

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$= - \int_0^{\infty} x \left(-\lambda e^{-\lambda x}\right) dx$$

$$\begin{cases} u = x & dv = -\lambda e^{-\lambda x} dx \\ du = dx & v = e^{-\lambda x} \end{cases}$$

$$= -\left(x e^{-\lambda x} \Big|_0^\infty - \int_0^\infty e^{-\lambda x} dx \right)$$

$$= -\left(\cancel{(0 - 0)} + \left(+\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty \right)$$

$$= -\left(\frac{1}{\lambda}(0-1)\right) = \boxed{\frac{1}{\lambda}} \checkmark$$

<u>note</u>:

- X has units $\frac{\text{time}}{\text{event}}$
- $E[X]$ " " $\frac{\text{time}}{\text{event}}$
- $\frac{1}{\lambda}$ " " $\frac{\text{time}}{\text{event}}$
- λ " " $\frac{\text{event}}{\text{time}}$

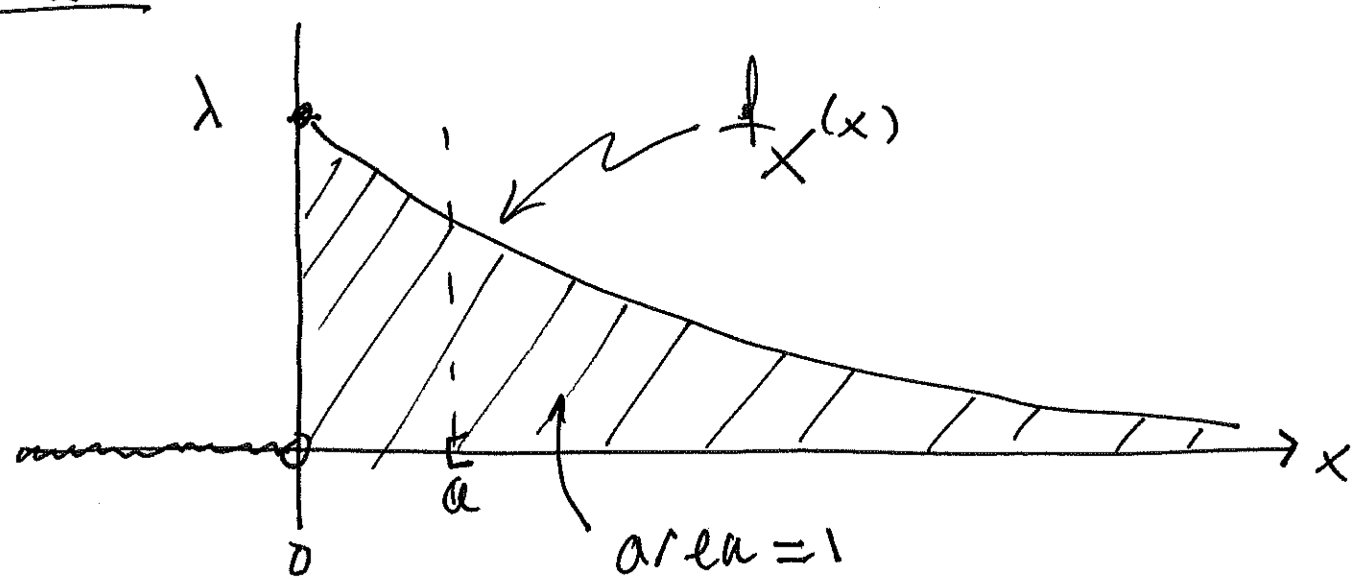
so λ is the <u>event rate</u>.

<u>Lemma</u>

If X is an Exponential r.v. then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

<u>Picture</u>



Proof

Let $a > 1$. Then

$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right)$$

$$= \boxed{e^{-\lambda a}}$$

∎

Ex.

The time X until the next earthquake (4.0 or above) is modeled as an Exponential r.v. with mean 53 minutes (world-wide).

What is the Prob. that such an earthquake occurs within the next 10 minutes?

i.e. find

$$P(0 \le X \le 10).$$

Solution

$$E[X] = \frac{1}{\lambda} = 53$$

$$\therefore \lambda = \frac{1}{53}$$

Thus

$$P(0 \le X \le 10) = P(X \ge 0) - P(X > 10)$$

$$= 1 - e^{-\frac{1}{53} \cdot 10}$$

$$= \boxed{0.1719}$$

Exercise

Find a such that the Prob. of an earthquake within next a minutes is 0.99

answer

$$a = 53 \cdot \ln(100)$$

$$= 244.07 \text{ min}$$

$$= 4.068 \text{ hrs}$$

3.2 Cumulative Distribution Functions (CDF)

Defn

Let X be a r.v. (disc. or cont.)

The CDF of X is

$$F_X(x) = P(X \leq x)$$

observe

- if X is discrete

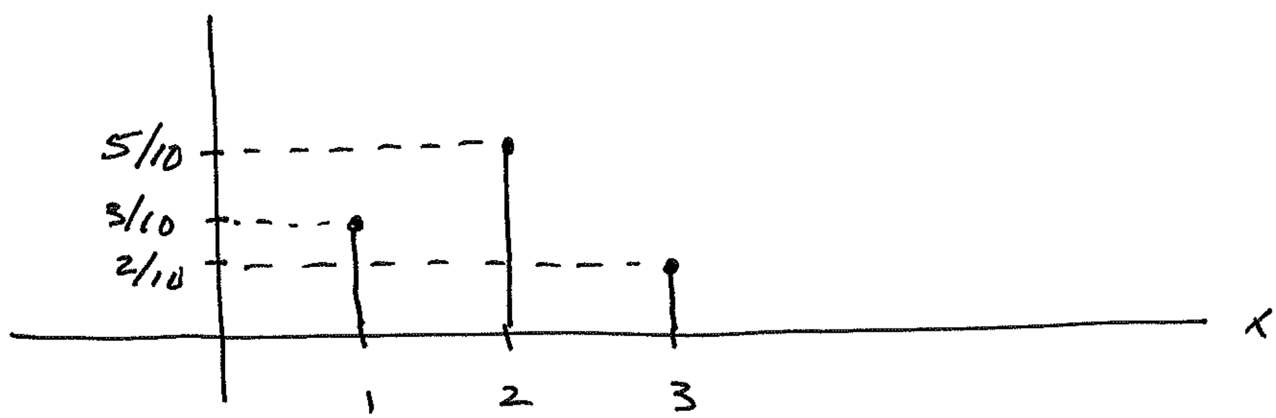
$$\boxed{F_X(x) = \sum_{\substack{k \le x \\ \uparrow \\ \text{in } \underline{\text{range}}(X)}} p_X(k)}$$

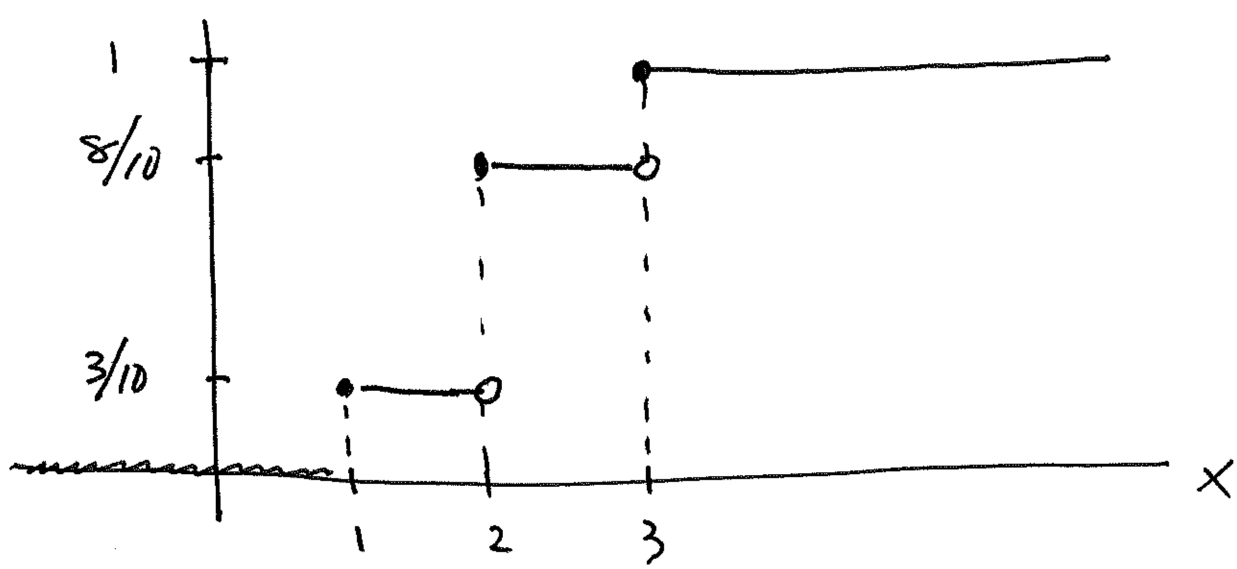
- if X is continuous

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt}$$

Ex. (discrete)

PMF: $$p_X(k) = \begin{cases} 3/10 & \text{if } k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$$





$\Rightarrow 0$

CDF : $F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ 3/10 & \text{if } 1 \le x < 2 \\ 8/10 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$

Ex. (continuous)

Cont. unif on $[1, 3]$

PDF $f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$