CSE 107 2-3-26

3.1 Continuous Random Variables

Defn

A r.v. $X : \Omega \to \mathbb{R}$ is called continuous iff there exists a function

$$f_X : \mathbb{R} \to \mathbb{R}$$

called the probability density function (PDF) of X, such that

$$P(X \in I) = \int_I f_X(x)\,dx$$

for 'any' subset $I \subseteq \mathbb{R}$. In particular

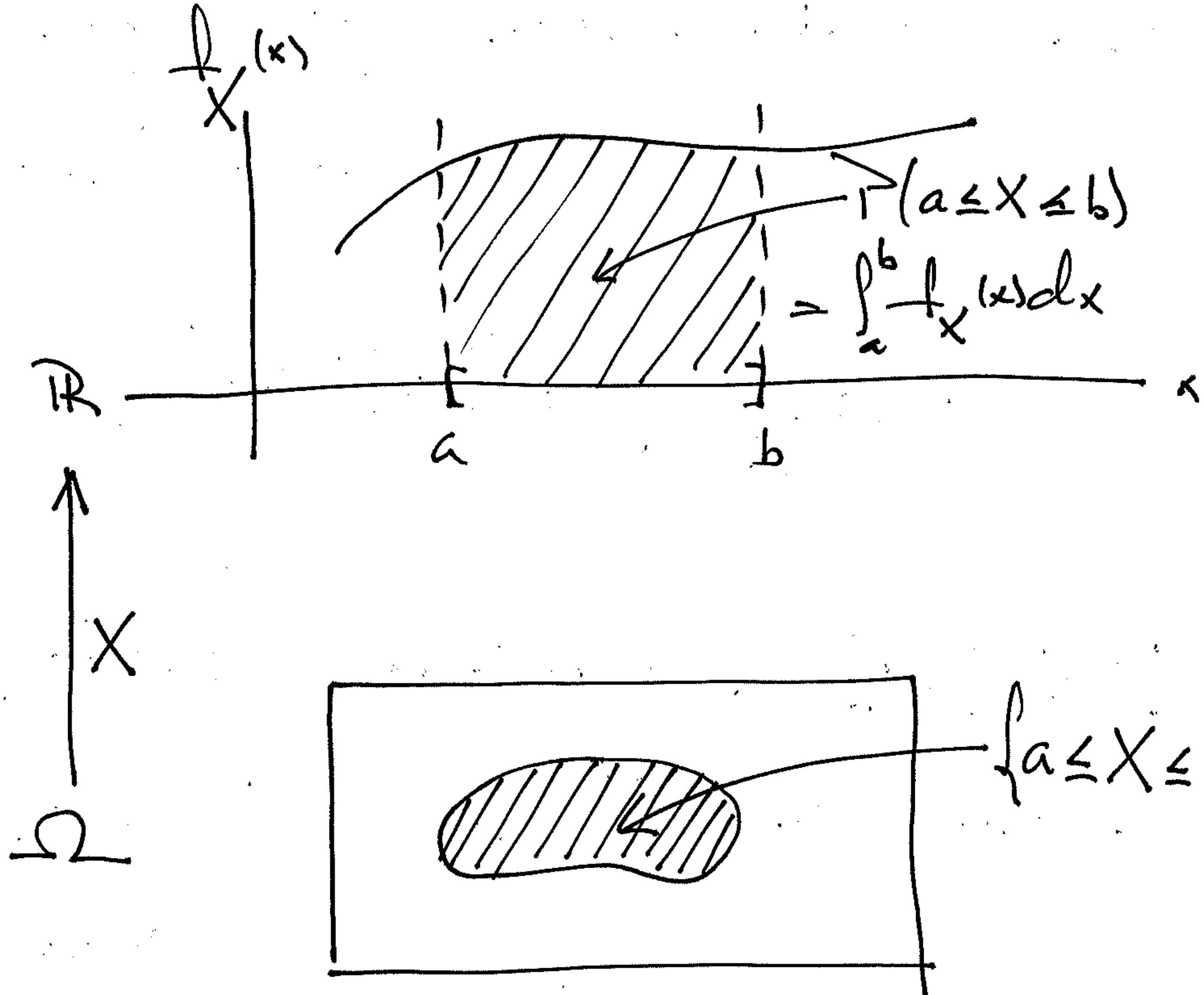
$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

<u>note</u>: integrating over $[a,b]$, $(a,b]$, $[a,b)$, (a,b) are all the same, since

$$\int_a^a f_X(x)\,dx = 0$$

also, necessarily

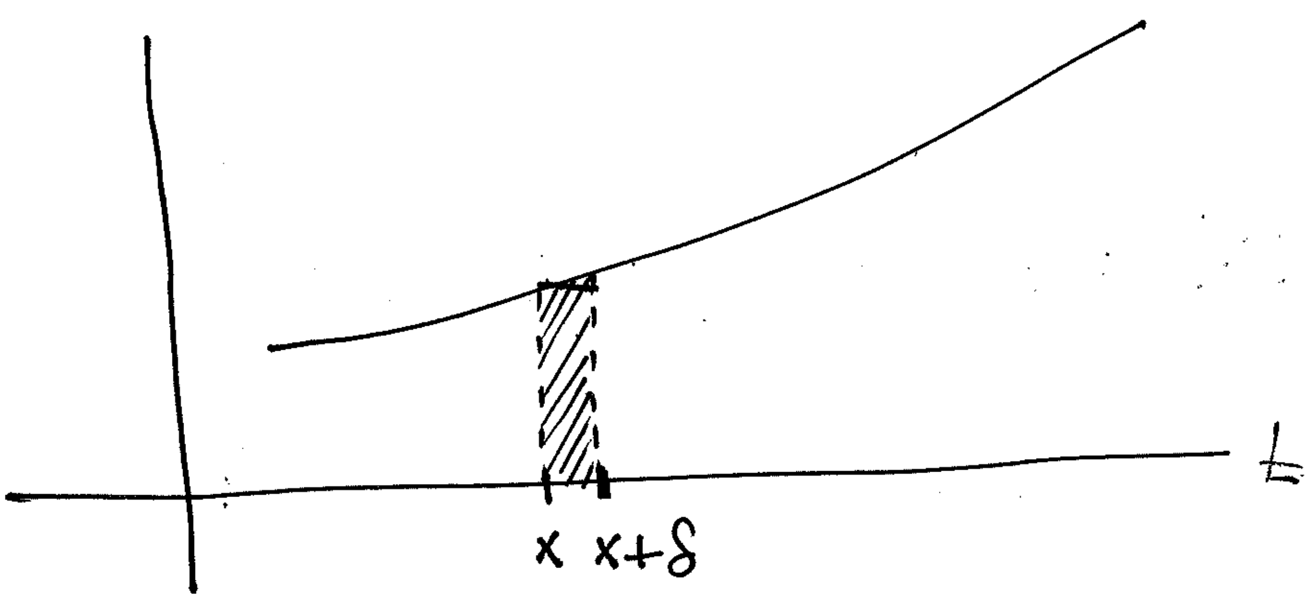
$$P(X=a) = \int_a^a f_X(x)\,dx = 0 .$$



<u>note</u>: PDF does not give 'mass' at a point. Indeed $P(X=a)=0$, always.

Instead $f_X(x)$ gives 'mass density' at x.

Let $f_X(t)$ be continuous at $x \in \mathbb{R}$



let $\delta > 0$ be small. Then $f_X(t)$ is approx. constant on $[x, x+\delta]$. Then

$$\underbrace{P(x \le X \le x+\delta)}_{\text{Probability or 'mass'}} = \int_x^{x+\delta} f_X(t)\,dt \approx \underbrace{f_X(x)}_{??} \cdot \underbrace{\delta}_{\text{length}}$$

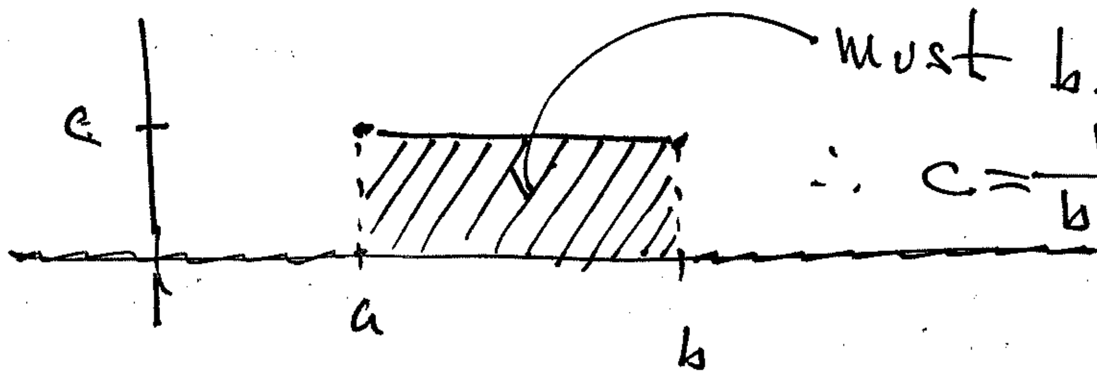
$\therefore$ $f_X(x)$ has units: $\underbrace{\dfrac{\text{Probability}}{\text{length}}}$

also called 'Probability density'

Note: The PDF can have isolated discontinuities.

Ex Continuous Uniform on $[a, b]$.

$$f_X(x) = \begin{cases} c & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



area must be $= 1$

$\therefore c = \dfrac{1}{b-a}$

necessarily

$$
\begin{aligned}
1 &= P(\Omega) \\
&= P(X \in \mathbb{R}) \\
&= P(-\infty < X < \infty) \\
&= \int_{-\infty}^{\infty} f_X(x)\,dx \\
&= \int_a^b c\,dx = c \cdot x \Big|_a^b = c(b-a)
\end{aligned}
$$

$$\therefore \quad c = \frac{1}{b-a}$$

Thus cont. unif on $[a,b]$ means

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

<u>Ex.</u> Piecewise Constant PDF.

Alice's driving time to work is 40–44 minutes in good weather and 44–50 minutes in bad weather, uniformly distributed in each case. Suppose

$$P(\text{good weather}) = \frac{2}{3}$$

find PDF of X, Alice's driving time.

Solution

we're given

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \le x \le 44 \\ c_2 & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

observe

$$\frac{2}{3} = P(\text{good}) = \int_{40}^{44} c_1\,dx = c_1 x \Big|_{40}^{44} = c_1(44-40)$$

$$\therefore \frac{2}{3} = 4 \cdot c_1 \Rightarrow c_1 = \frac{2}{3 \cdot 4} = \frac{2}{12} = \boxed{\frac{1}{6}}$$

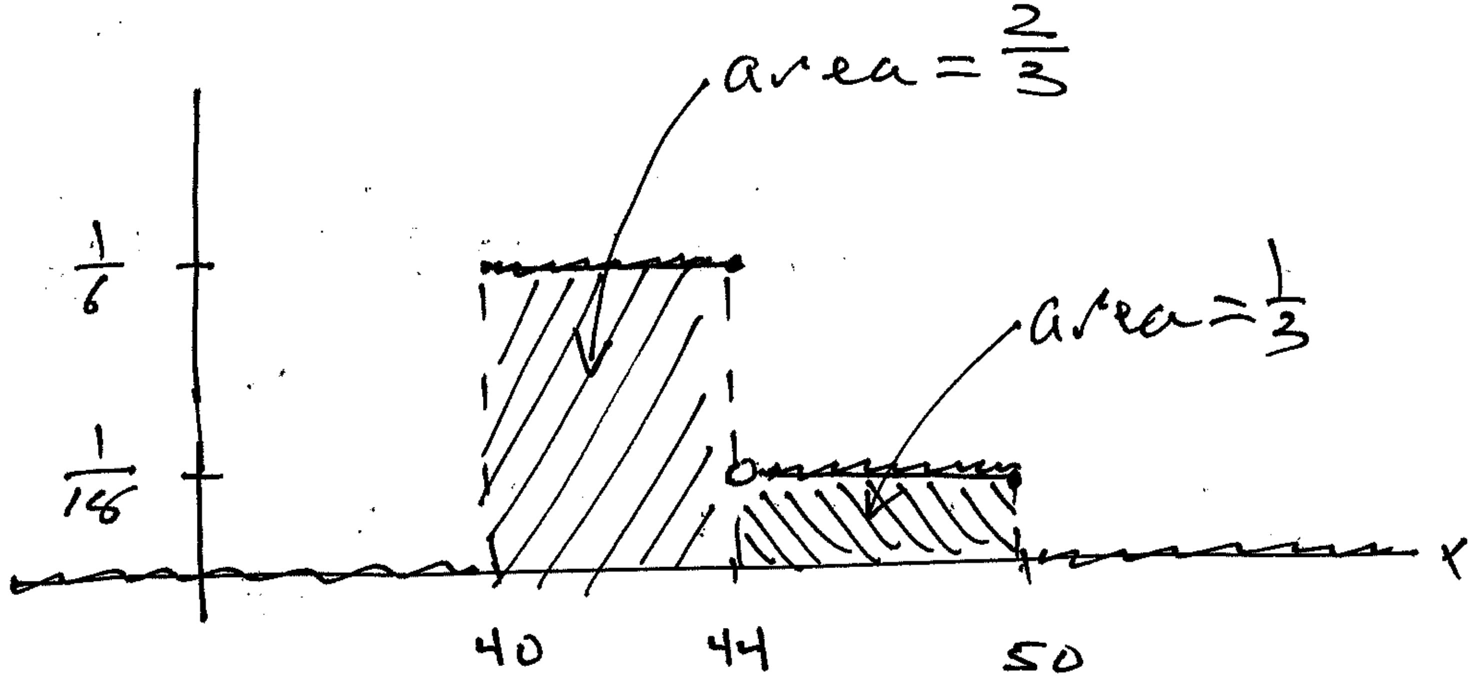
Also

$$\frac{1}{3} = 1 - P(\text{good}) = P(\text{bad})$$

$$= \int_{44}^{50} c_2 \, dx = c_2(50-44) = 6c_2$$

$$\therefore \quad c_2 = \boxed{\frac{1}{18}}$$

$$\therefore \quad f_X(x) = \begin{cases} 1/6 & \text{if } 40 \le x \le 44 \\ 1/18 & \text{if } 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$



<u>note</u>

- $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

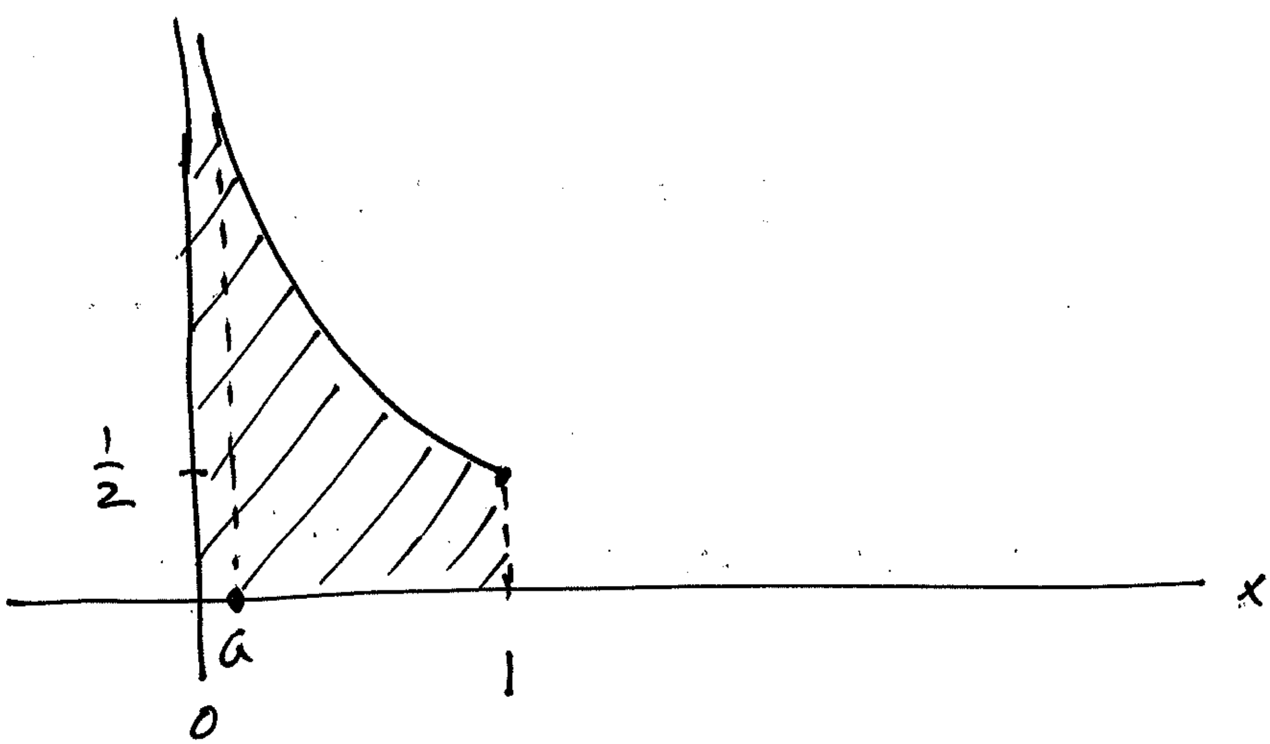
- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

for <u>any</u> PDF.

<u>note</u>: a PDF need not be bounded above.

<u>Ex</u>

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



observe

$$\int_0^1 \frac{1}{2\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \tfrac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{\cancel{2}} \lim_{a\to 0^+} \cancel{2}\cdot x^{1/2}\Big|_a^1$$

$$= \lim_{a\to 0^+} (1-\sqrt{a}) = 1$$

Defn

The expected value of a cont. r.v. is

$$\boxed{E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx}$$

i.e. the (approximate) average value of X over many trials of the experiment.

Expected Value rule

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx}$$

(see #4 on P. 185)

Defn

The n^{th} moment of X is

$$E[X^n] = \int_{-\infty}^{\infty} x^n \cdot f_X(x)\, dx$$

Also, the <u>variance of X</u> is

$$\boxed{Var(X) = E[(X - E[X])^2]}$$

<u>Theorem</u> (variance in terms of moments)

$$Var(X) = E[X^2] - E[X]^2$$

<u>Proof</u>

Let $\mu = E[X]$. Then

$$Var(X) = E[(X - \mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 f_X(x)\, dx}_{E[X^2]} - 2\mu \underbrace{\int_{-\infty}^{\infty} x f_X(x)}_{\underbrace{E[X]}_{\mu}} + \mu^2 \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{1}$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - (E[X])^2$$