CSE 107 2-20-26

Supplemental Lecture

Sums and convolutions

Let X, Y be independent, discrete r.v.s and let

$$Z = X + Y$$

Then

$$p_Z(z) = P(Z = z)$$

$$= P(X + Y = z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X = x, Y = y)$$

$$= \sum_{\{(x,z-x) \mid x \in \underline{\text{range}}(X)\}} P(X=x, Y=z-x)$$

$$= \sum_{x} P(X=x) \cdot P(Y=z-x)$$

$$= \underbrace{\sum_{x} p_X(x) \cdot p_Y(z-x)}_{\text{Convolution Product of } p_X \text{ and } p_Y}$$

i.e.

$$\boxed{(p_X * p_Y)(z) = \sum_{x} p_X(x)\, p_Y(z-x)}$$ defn and notation

Thus

$$p_{X+Y} = p_X * p_Y = p_Y * p_X$$

Now let X, Y be independent continuous r.v.s, and let

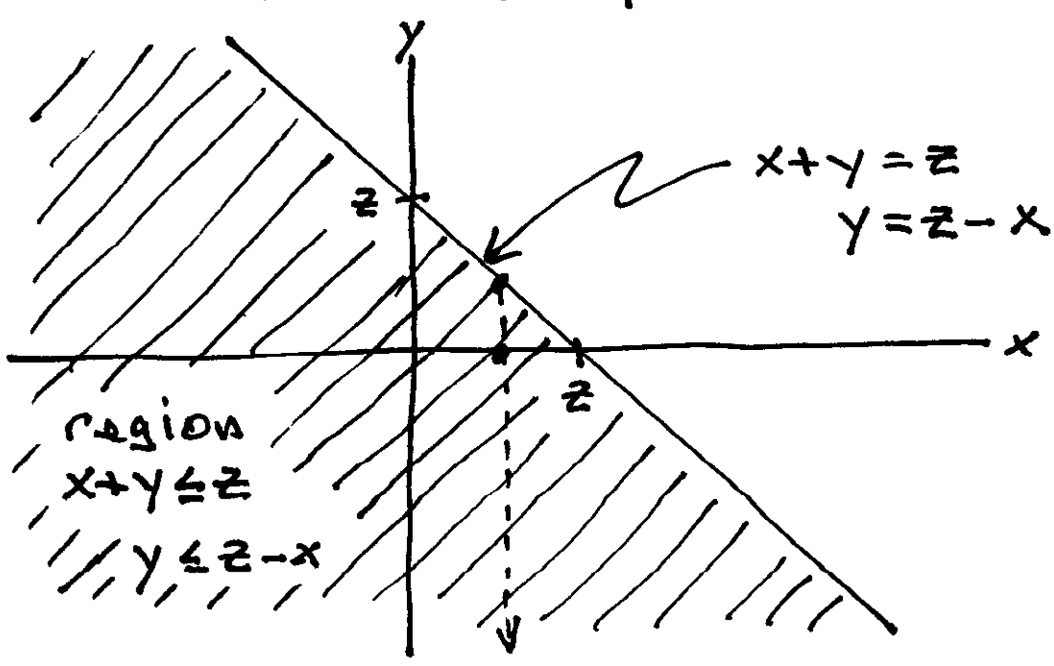
$$Z = X + Y$$

Then

$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= \iint_{\{(x,y) \mid x+y \le z\}} f_{X,Y}(x,y)\, dy\, dx$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, F_Y(z-x)\, dx$$

Differentiate w.r.t. z to get

$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x)\, F_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \frac{d}{dz} F_Y(z-x)\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx}_{\substack{\text{Convolution Product} \\ (f_X * f_Y)(z)}}$$

Thus

$$\boxed{f_Z(z) = (f_X * f_Y)(z)}$$

i.e.

$$f_{X+Y} = f_X * f_Y = f_Y * f_X$$

<u>Note</u>:

Both discrete and continuous formulae for distribution of $X+Y$ <u>require</u> X, Y to be independent.

Ex

let X, Y be indep. cont. unif. on $[a, b]$. Thus

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and similarly for Y. Let

$$Z = X + Y$$

Then

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

The integrand is $\frac{1}{(b-a)^2}$ iff

both

$$a \le x \le b \quad \text{and} \quad a \le z-x \le b$$

i.e.

$$a \le x \le b \quad \text{and} \quad z-b \le x \le z-a$$

(and 0 otherwise)

i.e. iff

$$\max(a, z-b) \le x \le \min(b, z-a)$$

Thus

$$f_Z(z) = \int_{\max(a,z-b)}^{\min(b,z-a)} \frac{1}{(b-a)^2}\, dx$$

$$= \frac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2}$$

The interval of integration is empty iff either

$$z - a < a \implies z < 2a$$

or

$$z - b > b \implies z > 2b$$

Thus we require $2a \le z \le 2b$, so

$$\boxed{f_Z(z) = \begin{cases} \dfrac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2} & \text{if } 2a \le z \le 2b \\ 0 & \text{otherwise} \end{cases}}$$

Exercise

Do same thing with X unif on $[a,b]$ and Y unif. on $[c,d]$

note: $\text{range}(\underbrace{X+Y}_{Z}) = [a+c, b+d]$

Ex.
Let X, Y be indep. normal r.v.s with

means: μ_X, μ_Y

and

variances: σ_X^2, σ_Y^2

Let $Z = X+Y$.

Thus

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma_X} e^{-(x-\mu_X)^2/2\sigma_X^2}$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y}\cdot e^{-(y-\mu_Y)^2/2\sigma_Y^2}$$

so

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X}\cdot e^{-(x-\mu_X)^2/2\sigma_X^2}\cdot\frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-(z-x-\mu_Y)^2/2\sigma_Y^2}\,dx$$

⋮ (exercise, see Wikipedia)

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{\sigma_X^2+\sigma_Y^2}}\cdot e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}}$$

$\therefore$ Z is normal with mean $\mu_X+\mu_Y$, Var $\sigma_X^2+\sigma_Y^2$

<u>Corollary</u>

$$X, Y \text{ normal} \Rightarrow aX, bY \text{ normal}$$

$$\Rightarrow aX + bY \text{ normal}.$$

i.e. any linear combination of normals is normal.

4.2 Covariance/correlation

Defn

The covariance of r.v.s X, Y is

$$\boxed{\mathrm{Cov}(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]}$$

Observe that

$$\mathrm{Cov}(X,X) = E\left[(X-E[X])^2\right]$$
$$= \mathrm{Var}(X)$$

Defn: if $\mathrm{Cov}(X,Y)=0$, we say that X, Y are uncorrelated.

- If $\mathrm{Cov}(X,Y) > 0$, we say X, Y are <u>positively correlated</u>.

- If $\mathrm{Cov}(X,Y) < 0$, we say X, Y are <u>negatively correlated</u>.

observe

$$\mathrm{Cov}(X,Y) = E\left[(X - \overset{E[X]}{\overset{\downarrow}{\mu_X}})(Y - \overset{E[Y]}{\overset{\downarrow}{\mu_Y}})\right]$$

$$= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y]$$

$$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y$$

$$- \mu_X \mu_Y - \cancel{\mu_X \mu_Y} + \cancel{\mu_X \mu_Y}$$

$$= E[XY] - E[X]E[Y].$$

Hence

$$\boxed{\operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y]}$$

Note

$$X, Y \text{ indep.} \Rightarrow E[XY] = E[X]E[Y]$$

$$\Rightarrow \operatorname{Cov}(X,Y) = 0$$

$$\Rightarrow X, Y \text{ uncorrelated.}$$

The converse is false however

see Ex. 4.13 on P. 218-219.

Recall: contrapositive of $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$ and is logically equivalent to $A \Rightarrow B$.

Thus

X, Y either Pos. or Neg. Correlated

$\Rightarrow$ X, Y not independent.