CSE 107 2-14-26

Supplemental Lecture

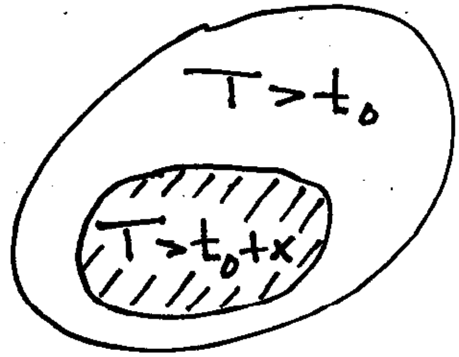
Recall (3.1) that

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

so for $x \geq 0$

$$P(X > x \mid A) = P(T > t_0 + x \mid T > t_0)$$

$$= \frac{P(T > t_0 + x, T > t_0)}{P(T > t_0)}$$



$$= \frac{P(T > t_0 + x)}{P(T > t_0)}$$

$$= \frac{e^{-\lambda(t_0 + x)}}{e^{-\lambda t_0}}$$

$$= \frac{\cancel{e^{-\lambda t_0}} \cdot e^{-\lambda x}}{\cancel{e^{-t_0}}}$$

$$= e^{-\lambda x}$$

so

$$F_{X|A}(x) = P(X \leq x \mid A)$$

$$= 1 - P(X > x \mid A)$$

$$= 1 - e^{-\lambda x}$$

$$F_{X|A}(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

now differentiate w.r.t. x

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Thus additional time to burn-out is also exponential, and Exponential r.v. has no memory.

<u>Exercise</u>

Do same thing for Geometric(p).

$$\overbrace{\underbrace{0\;0\;0\;0\cdots 0}_{m}\;\underbrace{0\;0\;0\cdots 0\;0\;1}_{X}}^{Y}$$

let Y = # trials to 1^{st} success

X = # trials after m^{th}, to 1^{st} success

$A = \{ Y > m \}$

show that

$$P_Y(k) = P_{X|A}(k) \quad (k=1,2,\dots \quad)$$

If $A_1, A_2, \dots, A_n \subseteq \Omega$ form a partition of Ω, total-Prob. thm gives

$$P(X \le x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \le x \mid A_i)$$

Thus

$$F_X(x) = \sum_{i=1}^{n} P(A_i)\, F_{X|A_i}(x)$$

differentiate w.r.t x !

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x)}$$

As in discrete case, we can condition on another r.v.

<u>Defn</u>

If X, Y are jointly continuous r.v.s and $f_Y(y) > 0$ for all $y \in \mathbb{R}$, the conditional PDF of X given Y is

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

<u>note</u>

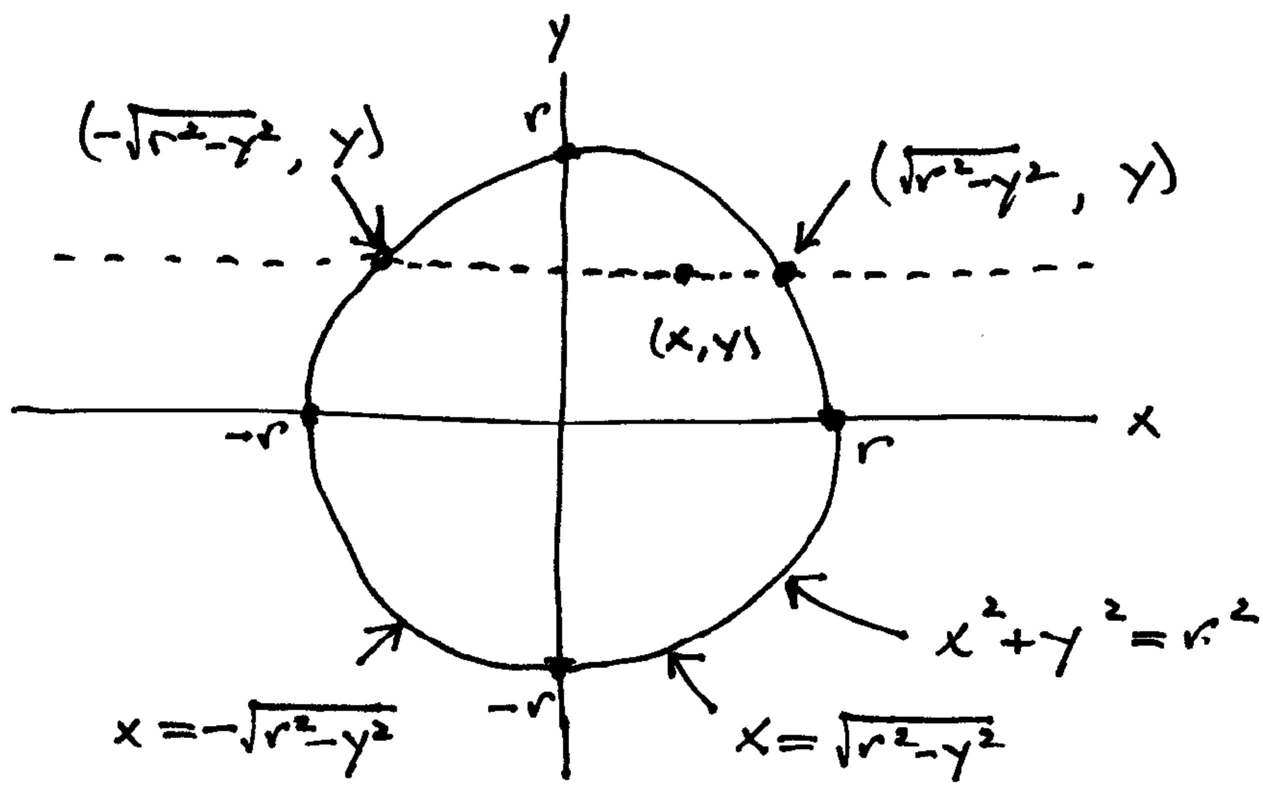
$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \frac{1}{f_Y(y)} \cdot f_Y(y) = 1 \quad \checkmark$$

<u>Ex</u> throw a dart at a circular target centered at $(0,0)$. Assume the impact point (X,Y) to be jointly uniform over target.

Find: $f_{X,Y}(x,y)$, $f_Y(y)$ and $f_{X|Y}(x|y)$



We are given

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{\pi r^2} & \text{if } x^2+y^2\le r^2\\ 0 & \text{otherwise}\end{cases}$$

Thus

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$=\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \cdot x \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}$$

$$= \frac{2\sqrt{r^2-y^2}}{\pi r^2}$$

i.e.

$$f_Y(y) = \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } -r \le y \le r \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: though $f_{X,Y}(x,y)$ is uniform, the marginal $f_Y(y)$ is not

<u>Exercise</u>: find marginal $f_X(x)$.

Finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \frac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Exercise

find $f_{Y|X}(y|x)$

note: Same identities hold for PDFs as for PMFs

$$f_{X|Y}(x|y) \cdot f_Y(y) = f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$$

Conditional Expectation

Let X, Y be jointly continuous, $A \subseteq \Omega$ with $P(A) > 0$.

Defn

- $E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx$

- $E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$

Expected value rules

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx$

- $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx$

Total Expectation Theorem

- if $A_1, A_2, \ldots, A_n$ form a partition of Ω, $P(A_i) > 0$ for all i,

$$(1) \quad \boxed{E[X] = \sum_{i=1}^{n} P(A_i) E[X \mid A_i]}$$

- similarly

$$(2) \quad \boxed{E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X \mid Y=y] \, dy}$$

Proof of (1)

Total prob. thm for PDFs:

$$f_X(x) = \sum_{i=1}^{n} P(A_i) \, f_{X|A_i}(x)$$

mult. by x and integrate

$$\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \cdot \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)\,dx$$

$$\therefore\ E[X] = \sum_{i=1}^{n} P(A_i) \int_{-\infty}^{\infty} x f_{X|A_i}(x)\,dx$$

$$\therefore\ E[X] = \sum_{i=1}^{n} P(A_i)\, E[X|A_i] \qquad \blacksquare$$

<u>Proof of (2)</u>

$$RHS = \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx \right) f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) f_Y(y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f_{X,Y}(x,y) \, dx \, dy$$

$$= E[X] = LHS \quad \blacksquare$$

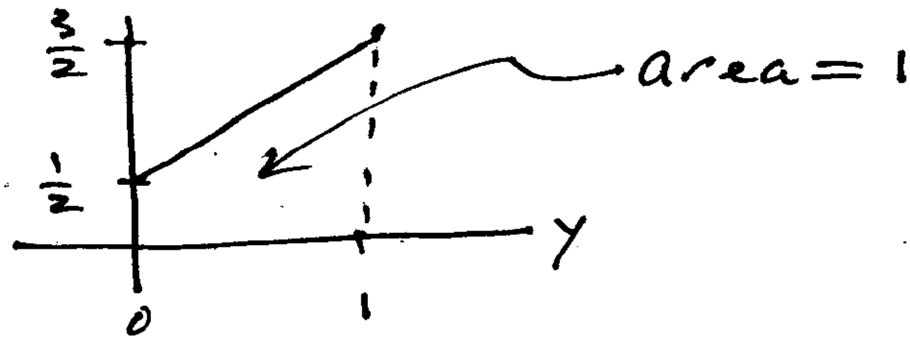
<u>note</u>: Partition in (2) is

$$A_y = \{Y = y\} \text{ for } y \in \text{range}(Y).$$

Ex.

Let X, Y be jointly cont. with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X]$.

Solution

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$$

$$= \int_0^1 x\left(\frac{x+y}{y+\frac{1}{2}}\right) dx = \frac{1}{y+\frac{1}{2}} \int_0^1 (x^2+xy)\, dx$$

$$= \frac{1}{y+\frac{1}{2}} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\Big|_0^1$$

$$= \frac{\frac{1}{2}y + \frac{1}{3}}{y+\frac{1}{2}} \quad (0 \le y \le 1)$$

so

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y)\, dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y + \frac{1}{3}}{\cancel{y + \frac{1}{2}}} \right) \cancel{(y + \tfrac{1}{2})}\, dy$$

$$= \frac{1}{2} \cdot \frac{1}{2} y^2 + \frac{1}{3} y \Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{3} = \boxed{\frac{7}{12}}$$

<u>check using:</u>

$$E[X] = \int_{-\infty}^{\infty} x \underbrace{f_X(x)\, dx}$$

first find this

<u>note</u> 2^{nd} version of Cond. expectation Thm.

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\, dy$$

can be written as

$$\boxed{E[X] = E\big[E[X|Y]\big]}$$

<u>Law of iterated Expectation</u>