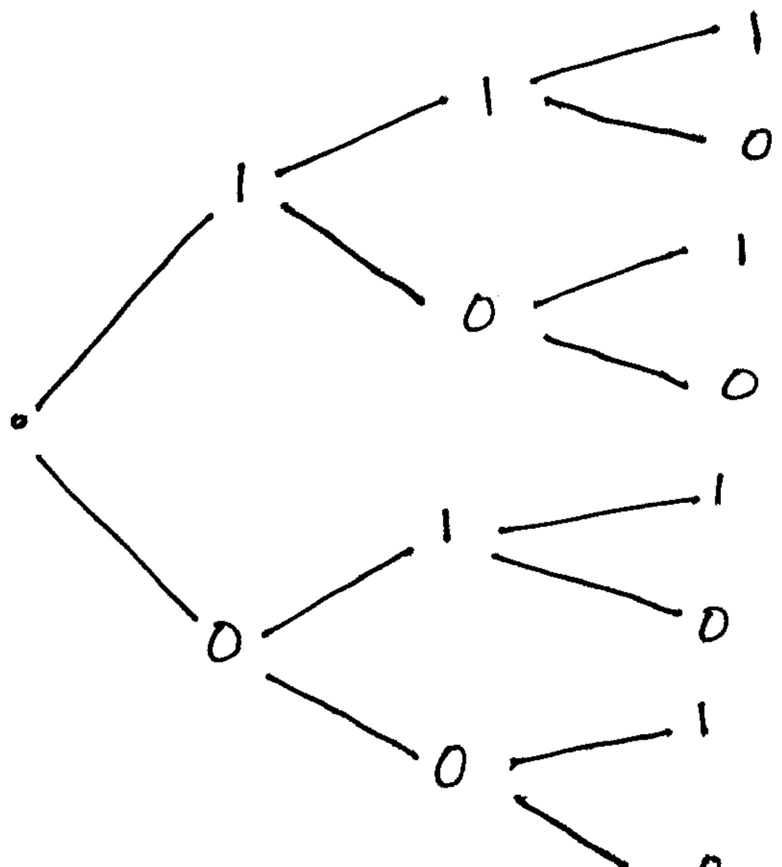
CSE 107 1-8-26

<u>Ex.</u> flip a fair coin 3 times

$\Omega = \{000, 001, 010, 011, 100, 101, 110, 111\}$

($H = 1$ and $T = 0$)



<u>outcome</u>	<u>Prob.</u>
111	1/8
110	1/8
101	1/8
100	1/8
011	1/8
010	1/8
001	1/8
000	1/8

so if $A = \{\text{exactly 1 head}\}$

$$= \{001, 010, 100\}$$

$$= \{001\} \cup \{010\} \cup \{100\}$$

Thus

$$P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Discrete Probability Law

If Ω is finite, then $P(\cdot)$ is determined by probabilities of singleton events $\{\omega\}$ for $\omega \in \Omega$.

so if $A \subseteq \Omega$ is

$$A = \{\omega_1, \omega_2, \ldots, \omega_k\}$$

then

$$P(A) = P(\omega_1) + P(\omega_2) + \cdots + P(\omega_k)$$

<u>Special case</u>

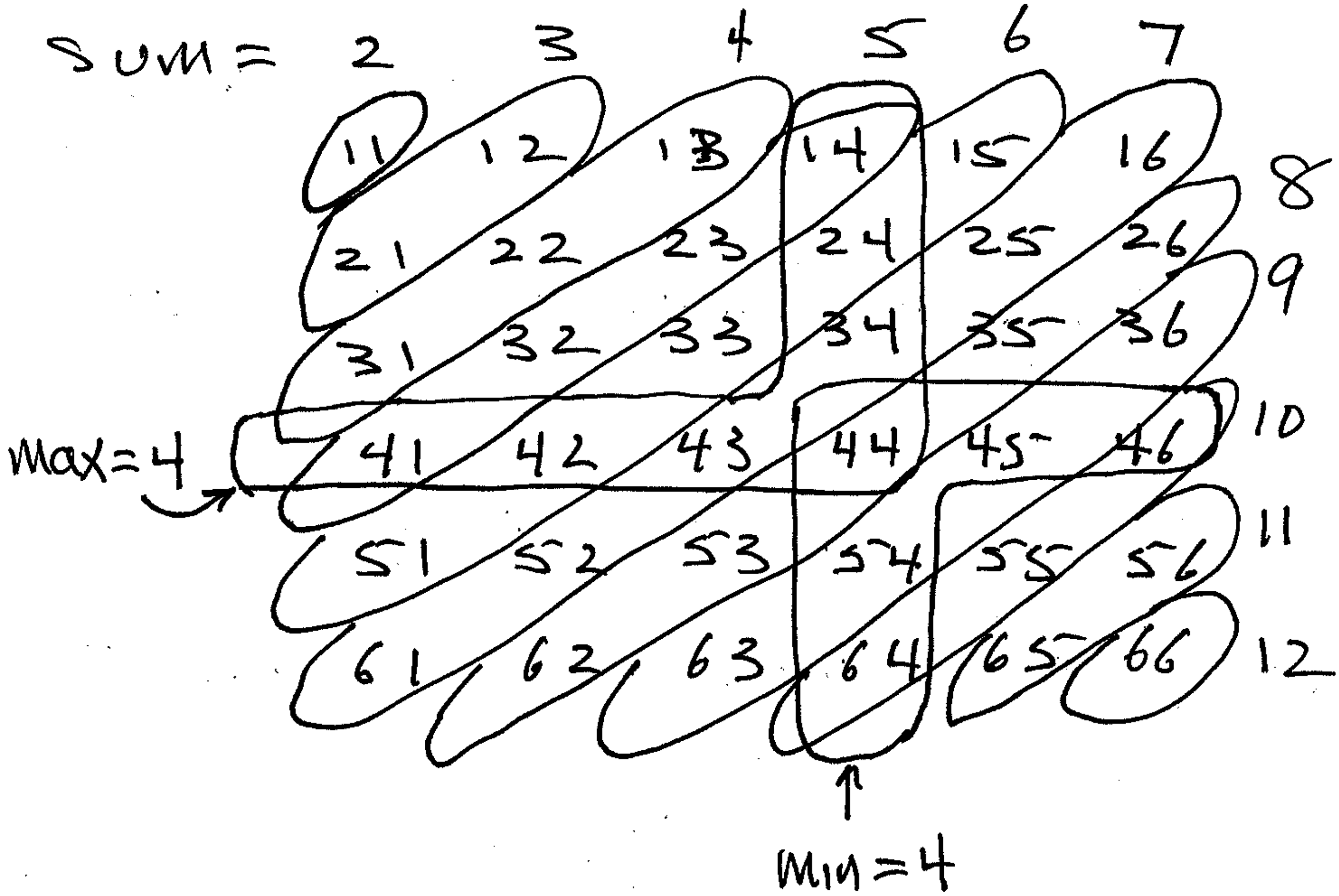
<u>Discrete Uniform Probability Law</u>

If Ω is finite and all $\omega \in \Omega$ are equally likely, then

$$P(A) = \frac{|A|}{|\Omega|}$$

Ex. 2 6-sided dice are thrown. Dice are fair, which means all outcomes are equally likely.

$$\Omega = \{(i,j) \mid 1 \le i \le 6 \text{ and } 1 \le j \le 6\}$$



$$P(\text{Sum}=2) = \frac{1}{36}$$

$$P(\text{Sum}=3) = \frac{2}{36}$$

$$\vdots$$

$$P(\text{Sum}=7) = \frac{6}{36}$$

$$\vdots$$

$$P(\text{Sum}=12) = \frac{1}{36}$$

$$P(\text{sum is even}) = \frac{1}{2}$$

$$P(\ldots\ \text{odd}) = \frac{1}{2}$$

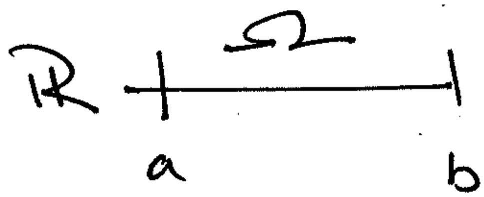
$$P(\max = 4) = \frac{7}{36}$$

$$P(\min = 4) = \frac{5}{36}$$

$$P(\max > 4) = \frac{20}{36}$$

continuous models

Ω is a 'continuum'



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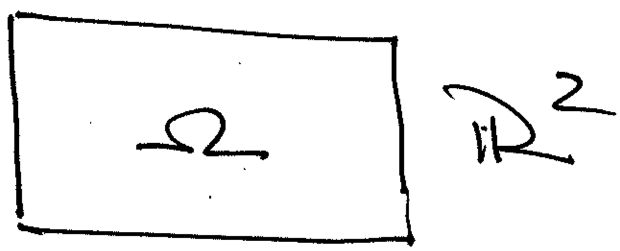

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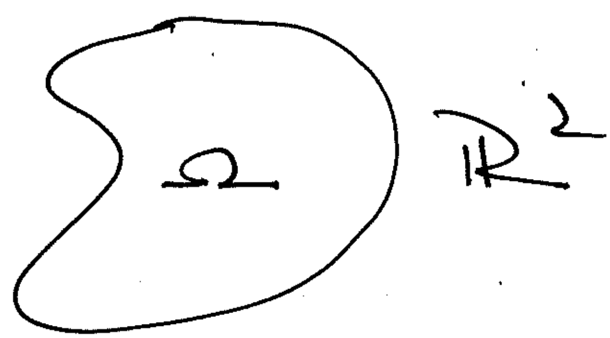
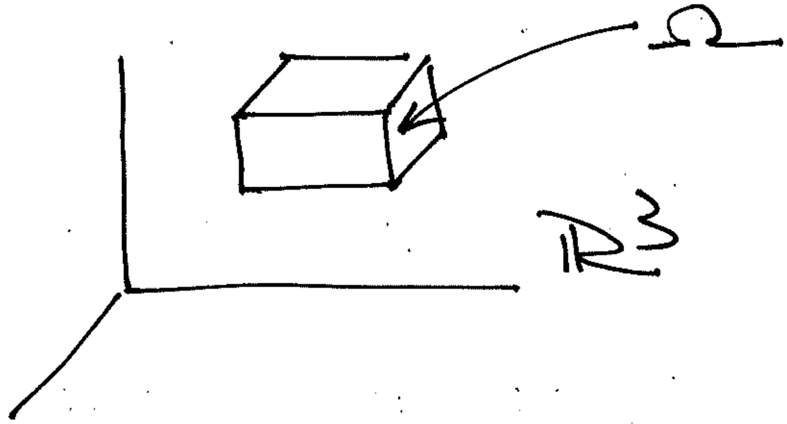
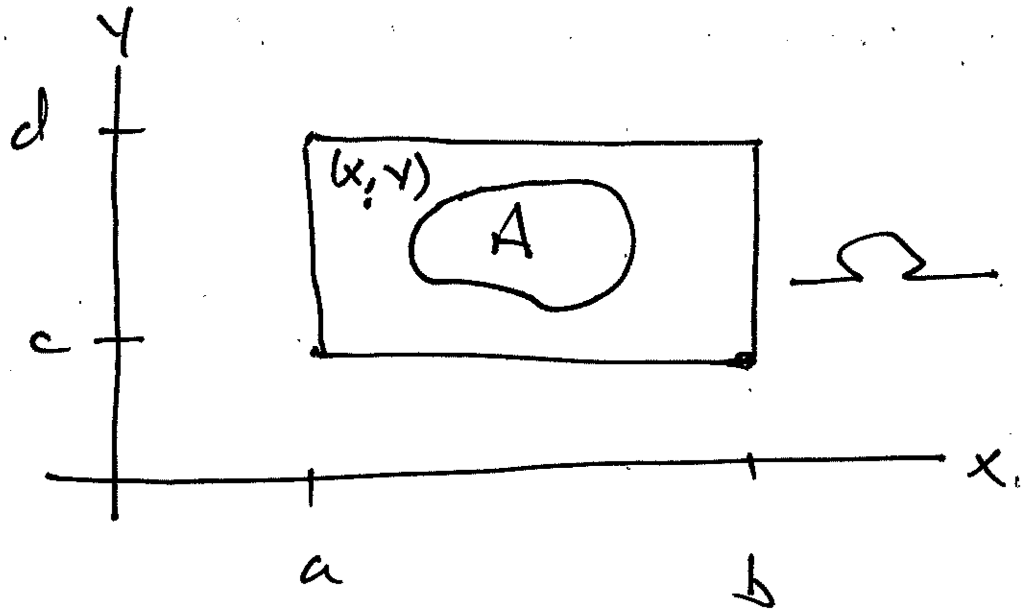




usually $P(\cdot)$ is a 'normalized' area.



$$P(A) = \text{const} \cdot \text{area}(A)$$

<u>note</u>:

$$1 = P(\Omega) = \text{const} \cdot \text{area}(\Omega)$$

$$\therefore \quad \text{const} = \frac{1}{\text{area}(\Omega)}$$

Therefore

$$P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

<u>continuous uniform Probability law</u>.

<u>note</u>

$$P((x,y)) = 0$$

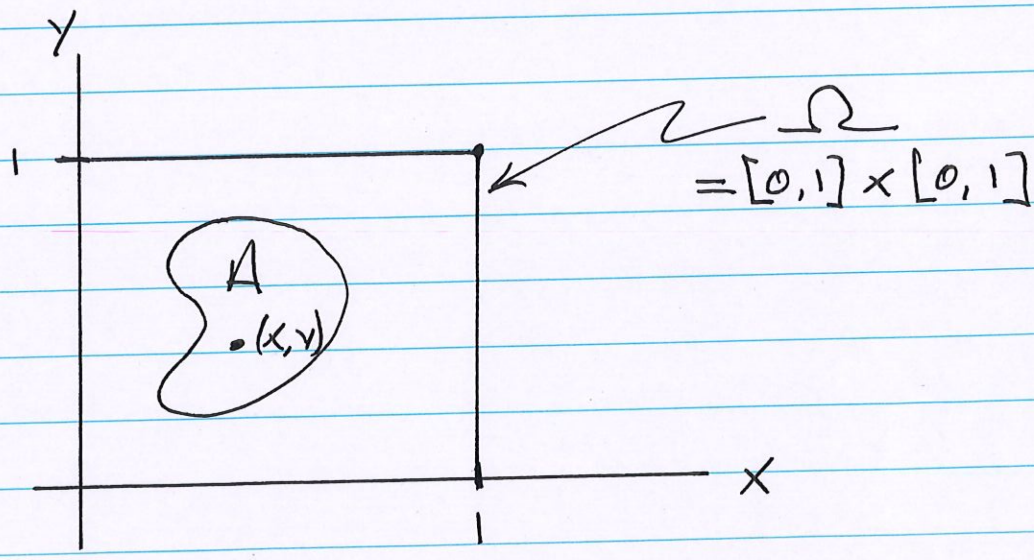
Ex.
R and J have a date, and each will be delayed by a random time in the range 0 to 1 hour. The 1st to arrive will wait 15 minutes, and leave if the other has not arrived.

What is the probability that they meet?

Let x be R's delay time and y be J's delay. Thus

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 1$$

We assume all delay pairs (x, y) are "equally likely".



We seek $P(M)$ where

$$M \subseteq \underbrace{[0,1] \times [0,1]}_{\Omega} \text{ .}$$

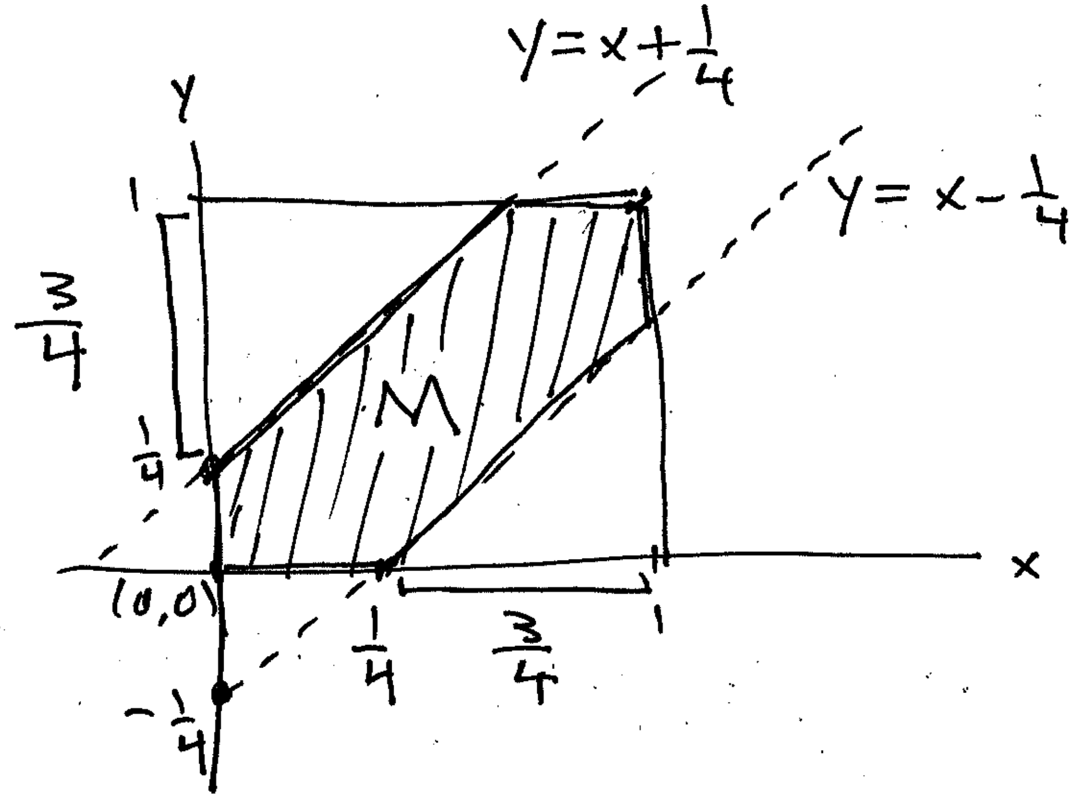
$$M = \left\{ (x,y) \,\middle|\, |x-y| \leq \tfrac{1}{4},\ (x,y) \in \Omega \right\}$$

observe $(x,y) \in M$ iff

$$|x-y| \leq \frac{1}{4}$$

$$\therefore \quad -\frac{1}{4} \leq x-y \leq \frac{1}{4}$$

$$\therefore \quad y \leq x + \frac{1}{4} \text{ and } y \geq x - \frac{1}{4}$$



$$P(M) = 1 - P(M^c)$$

$$= 1 - \left(\frac{3}{4}\right)^2$$

$$= 1 - \frac{9}{16}$$

$$= \boxed{\frac{7}{16}}$$

Exercise

Alice and Bob each pick a random in $[0,2]$. let Alice's be x, Bob's y. Find probability that both

$$2x > y^2 \text{ and } 2y \geq x^2$$

Properties of $P(\cdot)$

(a) if $A \subseteq B$, then $P(A) \leq P(B)$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

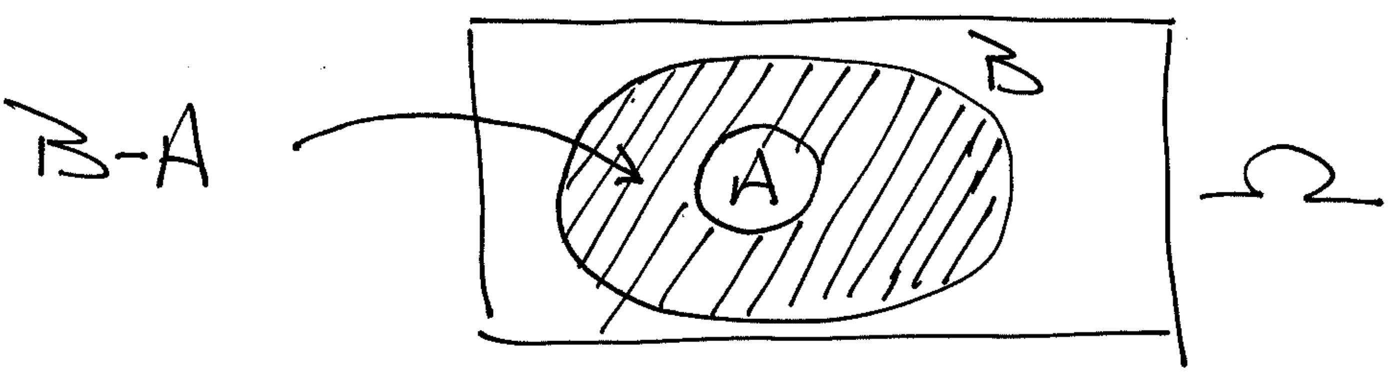
(c) $P(A \cup B) \leq P(A) + P(B)$

(d) $P(A \cup B \cup C) =$

$$P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$$

<u>Proof of (a)</u>

$$A \subseteq B \Rightarrow \begin{cases} B = A \cup (B - A) \\ A \cap (B - A) = \phi \end{cases}$$

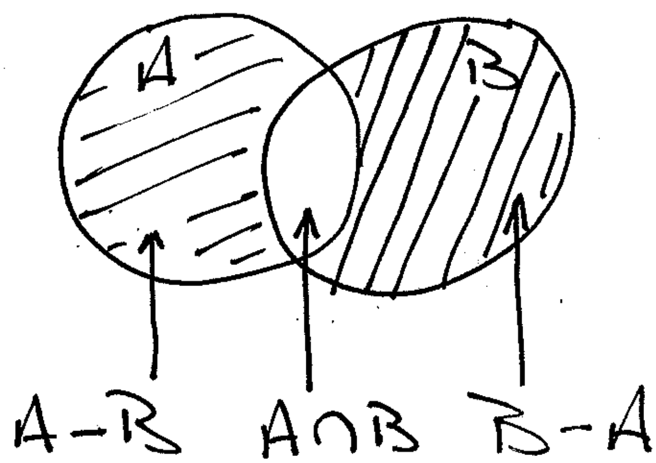


By axiom (iii)

$$P(B) = P(A) + P(B-A) \geq P(A)$$

$\therefore$ $P(A) \leq P(B)$. ■

<u>Proof of (b)</u>



By axiom (iii)

$$P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$$

<u>note</u> :

$$P(B) = P(B-A) + P(A \cap B)$$

$$\therefore P(B-A) = P(B) - P(A \cap B)$$

replace B by A:

$$P(A-B) = P(A) - P(A \cap B)$$

Thus

$$P(A \cup B) = \left[P(A) - \cancel{P(A \cap B)}\right] + \cancel{P(A \cap B)}$$

$$+ \left[P(B) - P(A \cap B)\right]$$

$$= P(A) + P(B) - P(A \cap B)$$

■