CSE 107 1-29-26

## Variance of Binomial$(n,p)$

If $X$ is Binomial$(n,p)$, then

$$X = X_1 + X_2 + \cdots + X_n$$

where $X_i$ = Bernoulli$(p)$, are independent. Thus

$$Var(X) = Var(X_1 + \cdots + X_n)$$

$$= Var(X_1) + \cdots + Var(X_n)$$

$$= \underbrace{p(1-p)+p(1-p)+\cdots+p(1-p)}_{n}$$

$$= np(1-p).$$

<u>Ex.</u> suppose we have weighted coin with $P(\text{head}) = p$, but we don't know $p$.

we compute <u>sample mean</u>

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

for each $n \geq 1$. (also relative frequency) note

$$E[X_i] = p$$

so

$$E[S_n] = E\left[\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n\right]$$

$$= \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \cdot \frac{1}{n} \cdot p$$

$$= \frac{1}{n} \sum_{i=1}^{n} p = \frac{1}{\cancel{n}} \cdot \cancel{n}p = p$$

so $S_n$ is an estimate of $E[S_n]$ $= p$. how good is the estimate?

note

$$\operatorname{Var}(\delta_n) = \operatorname{Var}\left(\sum_{i=1}^{n} \frac{1}{n} X_i\right)$$

$$= \sum_{i=1}^{n} \operatorname{Var}\left(\frac{1}{n} X_i\right)$$

$$= \sum_{i=1}^{n} \frac{1}{n^2} \operatorname{Var}(X_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} p(1-p) = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} p(1-p)$$

$$= \frac{p(1-p)}{n} \longrightarrow 0$$

Exercise

If $Var(Y) = 0$, then $Y = \text{const.}$

note

even if $X_i$ are not Bernoulli, same calculation shows

$$Var(S_n) = \frac{Var(X)}{n} \qquad X_i$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$