CSE 107 1-23-26

Supplemental Lecture

Expected Value rule

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y)$$

Ex $g(x,y) = ax + by + c$

show

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Proof.

$$E[aX + bY + c] = \sum_x \sum_y (ax + by + c)\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y ax\, p_{X,Y}(x,y) + \sum_x \sum_y by\, p_{X,Y}(x,y)$$

$$+ \sum_x \sum_y c\, p_{X,Y}(x,y)$$

$$= a \sum_x x \cdot \sum_y p_{X,Y}(x,y) + b \sum_y y \sum_x p_{X,Y}(x,y)$$

$$+ c \sum_x \sum_y p_{X,Y}(x,y)$$

$$= a \sum_x x\, p_X(x) + b \sum_y y\, p_Y(y) + c$$

$$= a\, E[X] + b\, E[Y] + c$$

$\blacksquare$

Ex (Previous)

$$E[X+3Y] = E[X] + 3E[Y]$$

$$= \left(1\cdot\frac{4}{20} + 2\cdot\frac{5}{20} + 3\cdot\frac{6}{20} + 4\cdot\frac{5}{20}\right)$$

$$+ 3\left(1\cdot\frac{6}{20} + 2\cdot\frac{9}{20} + 3\cdot\frac{5}{20}\right)$$

$$= \cdots = \boxed{\frac{169}{20}}$$

Multiple random variables

Given X, Y, Z, the joint PMF

$$p_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

have 3 2-variable Marginal PMFs

$$P_{X,Y}(x,y) = \sum_z P_{X,Y,Z}(x,y,z)$$

$$P_{X,Z}(x,z) = \sum_y P_{X,Y,Z}(x,y,z)$$

$$P_{Y,Z}(y,z) = \sum_x P_{X,Y,Z}(x,y,z)$$

can also get

$$P_X(x) = \sum_y \sum_z P_{X,Y,Z}(x,y,z)$$

$$P_Y(y) = \quad \cdots$$

$$P_Z(z) = \quad \cdots$$

Expected value rule

$$E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z)\, P_{X,Y,Z}(x,y,z)$$

suppose we have: $X_1, X_2, \ldots, X_n$

Then

$$E[a_1X_1 + a_2X_2 + \cdots + a_nX_n + b]$$

$$= a_1E[X_1] + a_2E[X_2] + \cdots + a_nE[X_n] + b$$

Mean & Variance of Binomial(n, p)

Recall

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (k = 0, 1, 2, \ldots n)$$

Then

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Easy way: note $X = X_1 + X_2 + \cdots + X_n$

where each X_i is Bernoulli(p).

Recall $E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$, so

$$\begin{aligned} E[X] &= E[X_1 + X_2 + \cdots + X_n] \\ &= E[X_1] + E[X_2] + \cdots + E[X_n] \\ &= p + p + \cdots + p = \boxed{np} \end{aligned}$$

for variance, will need discussion

Problem 2.4 #22 on p. 123.

$$Var(X) = E[X^2] - E[X]^2$$

answer

$$Var(X) = np(1-p)$$

<u>Proof later</u>.

<u>Ex.</u> <u>The Hat Problem</u>

suppose n people throw their hats into a box, then everyone gets a random hat back. (i.e. all $n!$ permutations are equally likely) have a random assignment

$$\{\text{hats}\} \to \{\text{people}\}$$

Let $X = \#$ people who get their own hat back. Find $E[X]$.

observe that $X = X_1 + X_2 + \cdots + X_n$

where each

$$X_i = \begin{cases} 1 & \text{if person } i \text{ gets own hat} \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: each permutation equally likely implies $P(X_i = 1) = \frac{1}{n}$, i.e. X_i is Bernoulli$(\frac{1}{n})$.

Thus $E[X_i] = \frac{1}{n}$ and

$$
\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n] \\
&= \frac{1}{n} + \cdots + \frac{1}{n} \\
&= n \cdot \frac{1}{n} = \boxed{1}
\end{aligned}
$$

2.6 Conditioning

Recall: Given $X: \Omega \to \mathbb{R}$, we defined

$$p_X(x) = \mathbb{P}(X = x)$$

note: we can use any Probability law. This includes conditional Prob. laws $\mathbb{P}(\cdot \mid A)$ where $A \subseteq \Omega$ and $\mathbb{P}(A) > 0$.

Defn

given $X: \Omega \to \mathbb{R}$, $A \subseteq \Omega$, $\mathbb{P}(A) > 0$, the conditional PMF of X given A is

$$p_{X|A}(x) = \mathbb{P}(\{X = x\} \mid A) = \frac{\mathbb{P}(\{X = x\} \cap A)}{\mathbb{P}(A)}$$

<u>note</u> the events $\{X=x\}\cap A$ are disjoint for different $x \in \underline{range}(X)$

so by total prob. thm.

$$P(A) = \sum_x P(\{X=x\}\cap A)$$

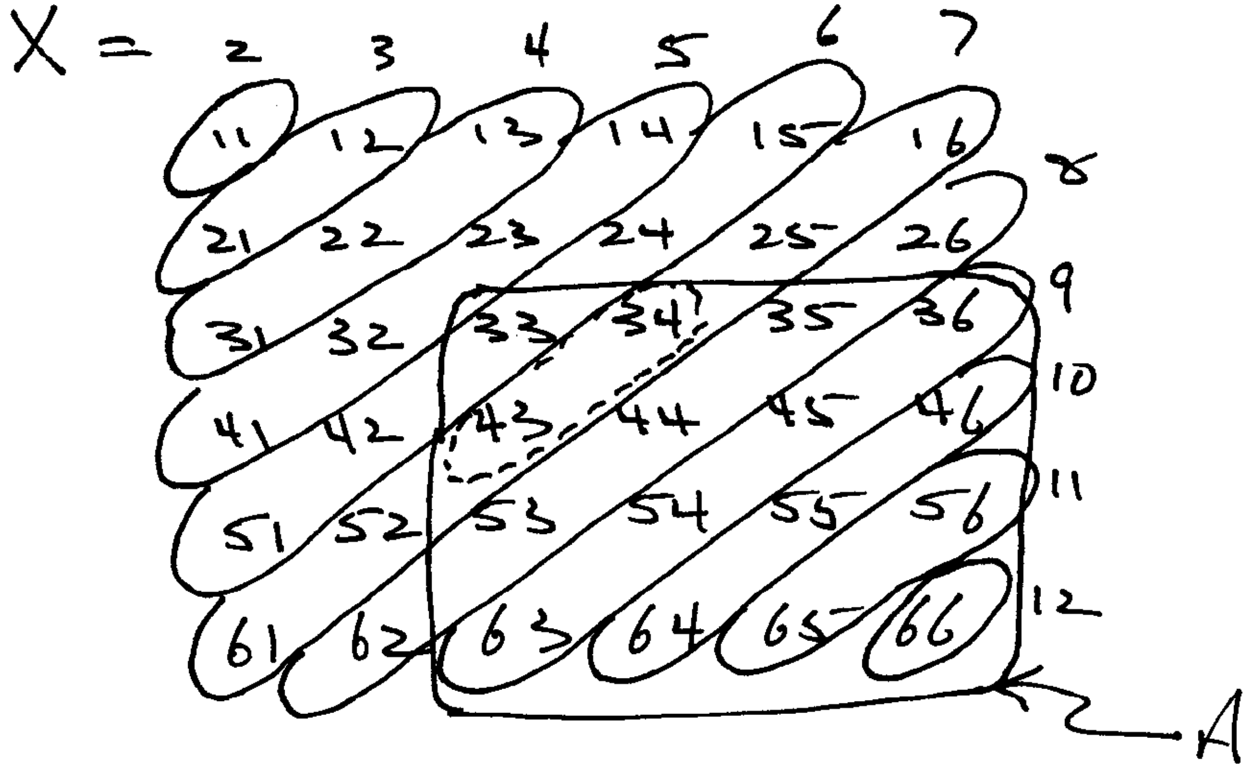
Thus

$$\sum_x P_{X|A}(x) = \frac{\sum_x P(\{X=x\}\cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)} = 1$$

so $P_{X|A}(x)$ is a valid PMF.

Ex. roll 2 fair, indep. 6-sided dice, let X = sum of 2 dice



PMF

$$P_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & k = 3, 11 \\ 3/36 & k = 4, 10 \\ 4/36 & k = 5, 9 \\ 5/36 & k = 6, 8 \\ 6/36 & k = 7 \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{\min \geq 3\}$, so $P(A) = \frac{16}{36}$.

Thus

$$P_{X|A}(7) = P(X=7 \mid A)$$

$$= \frac{P(\{X=7\} \cap A)}{P(A)} = \frac{2/36}{16/36} = \frac{2}{16}$$

and

$$P_{X|A}(k) = \begin{cases} 1/16 & \text{if } k = 6, 12 \\ 2/16 & \text{if } k = 7, 11 \\ 3/16 & \text{if } k = 8, 10 \\ 4/16 & \text{if } k = 9 \\ 0 & \text{otherwise} \end{cases}$$

Ex

Flip a weighted coin with $P(H) = p$ until the first head. Let X be the # of flips. Then X is Geometric(p). Recall

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{1^{st}$ head is within n flips$\}$, then

$$P(A) = \sum_{k=1}^{n} P_X(k)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p \qquad \{ k \leftarrow k+1$$

$$= p \sum_{k=0}^{n-1} (1-p)^k$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1}-(\cancel{1}-\cancel{p})} = 1-(1-p)^n$$

another way

$$P(A) = 1 - P(A^c)$$

$$= 1 - P(1^{st}\ n \text{ flips are tails})$$

$$= 1-(1-p)^n$$

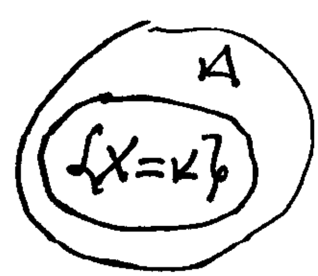
Thus

$$P_{X|A}(k) = P(\{X=k\}|A)$$

$\emptyset$ if $k>n$

$\{X=k\}$ if $1 \le k \le n$

$$= \frac{P(\{X=k\} \cap A)}{P(A)}$$



$$= \frac{P(X=k)}{P(A)}$$

$$= \frac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} \quad (\text{for } 1 \le k \le n)$$

so

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} & \text{if } k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

<u>Exercise</u> $\sum_{k=1}^{n} P_{X|A}(k) = 1$ <u>check</u>

Now let Y be another r.v. with

$$p_Y(y) = P(Y=y) > 0$$

for all $y \in \text{range}(Y)$. We specialize A to be $A = \{Y=y\}$

Defn
The conditional PMF of X given Y is

$$p_{X|Y}(x|y) = P(X=x \mid Y=y)$$
$$= \frac{P(X=x, Y=y)}{P(Y=y)}.$$

observe

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

also

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)}$$

Ex The Professor Problem

A professor answers questions wrongly with prob. $\frac{1}{4}$, indep. of other questions. The # of questions asked is 0, 1, or 2 with equal probability, $\frac{1}{3}$. Let

X = # questions asked

Y = # questions answered wrongly

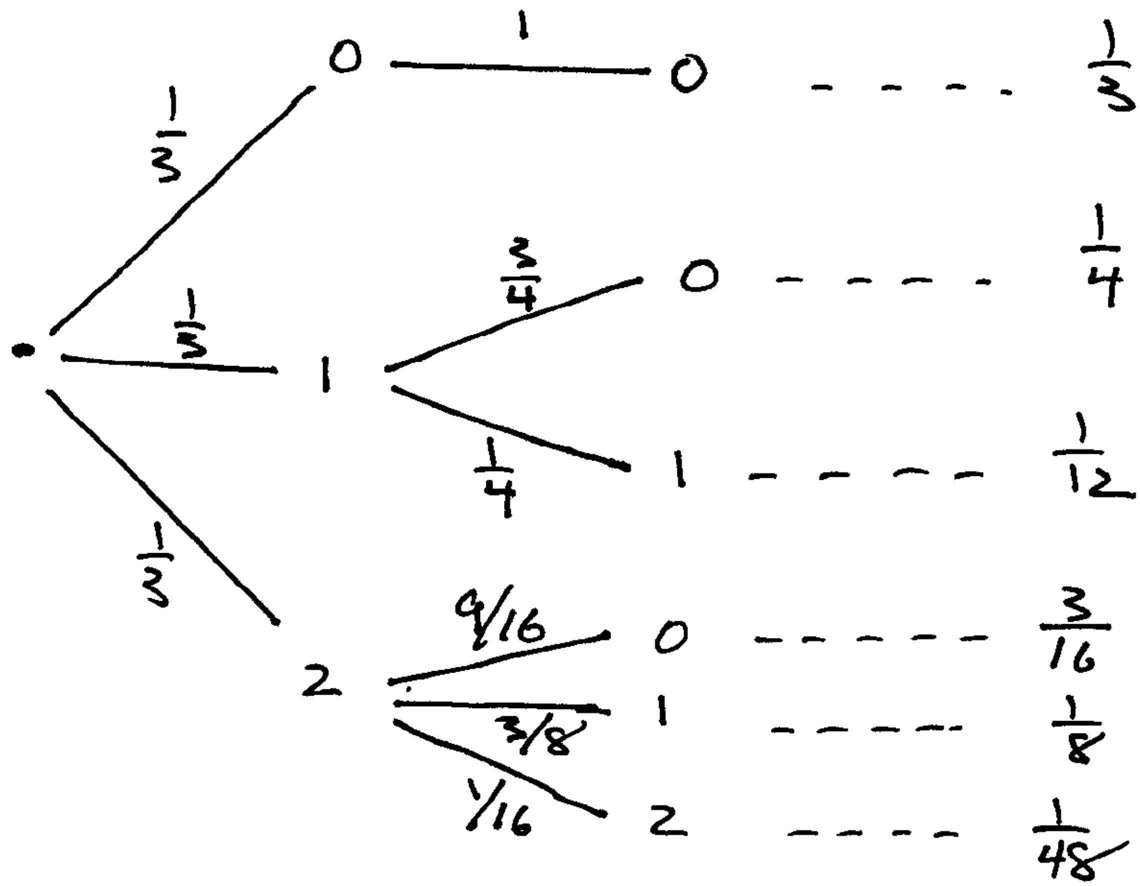
Find $P_{X,Y}(x,y)$.

Solution We are given

$$P_X(x) = \begin{cases} 1/3 & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

tree diagram

X #asked	Y #wrong	Probability



Both right $= \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

Both wrong $= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

1 right, 1 wrong $= \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{6}{16} = \frac{3}{8}$

$P_X(x)$

x	0	1	2
$P_X(x)$	1/3	1/3	1/3

$P_{Y|X}(y|x)$

y \ x	0	1	2
2			1/16
1		1/4	3/8
0	1	3/4	9/16

$P_{X,Y}(x,y)$

y \ x	0	1	2	$P_Y(y)$
2			1/48	1/48
1		1/12	1/8	10/48
0	1/3	1/4	3/16	37/48

<u>note</u> $$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

so

$p_{X|Y}(x|y)$

$y \backslash x$	0	1	2
2			1
1		2/5	3/5
0	16/37	12/37	9/37

note: see summary of facts on p. 103.

Conditional Expectation

Let X be a r.v., $A \subseteq \Omega$, and assume $P(A) > 0$.

Defn

$$\boxed{E[X|A] = \sum_x x\, p_{X|A}(x)}$$

conditional expectation of X given A.

Expected value rule:

$$\boxed{E[g(X)|A] = \sum_x g(x)\cdot p_{X|A}(x)}$$

can condition on $A = \{Y = y\}$ provided $p_Y(y) = P(Y = y) > 0$.

$$\boxed{E[X \mid Y=y] = \sum_x x\, p_{X|Y}(x|y)}$$

Total Expectation Theorem

If $A_1, A_2, \ldots, A_n$ form a partition of Ω, then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

Proof.

by the tot. Prob. Thm.

$$(*) \quad p_X(x) = \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

i.e.

$$P(X=x) = \sum_{i=1}^{n} P(A_i) \cdot P(X=x \mid A_i)$$

multiply both sides of * by x and sum over x to obtain

$$E[X] = \sum_x x\, p_X(x)$$

$$= \sum_x x \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

$$= \sum_x \sum_{i=1}^{n} P(A_i)\, x\, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \sum_x x\, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]$$

• see similar identities on
P. 104