CSE 107 1-22-26

- mid 1: Th. 1/29/26

  Exam: 7:10 - 8:15

  Break: 5 min

  lecture: 25 min.

Continue ...

disc. unif (a, b)

$$E[X] = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1}$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \right)$$

$$= \frac{1}{b-a+1} \left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2} \right)$$

$$= \underbrace{\cdots\cdots\cdots}_{\text{exercise}} = \boxed{\frac{a+b}{2}}$$

To get Var(X), Let

$$Y = X + (1-a)$$

so $Var(X) = Var(Y)$. let $n = b-a+1$ and note that Y is uniformily distributed on $\{1, 2, 3, \ldots, n\}$.

$$X = a \Rightarrow Y = a+(1-a) = 1$$

$$X = b \Rightarrow Y = b+(1-a) = n$$

we have $E[Y] = \frac{n+1}{2}$. need $E[Y^2]$.

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \left(\frac{\cancel{n}(n+1)(2n+1)}{6} \right)$$

$$= \frac{(n+1)(2n+1)}{6}$$

so

$$Var(Y) = E[Y^2] - E[Y]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \cdots = \frac{(n+1)(n-1)}{12} = \frac{(b-a)(b-a+2)}{12}$$

Ex. Poisson(λ)

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad (k = 0, 1, \ldots)$$

We will show

(1) $E[X] = \boxed{\lambda}$ ✓

(2) $E[X^2] = \lambda^2 + \lambda$ ✓

Hence

$$Var(X) = E[X^2] - E[X]^2$$

$$= (\lambda^2 + \lambda) - \lambda^2 = \boxed{\lambda}$$

Proof of (1)

$$E[X] = \sum_{k=\cancel{0}1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \qquad (*)$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \qquad \left(\begin{array}{c}\text{replace} \\ k \leftarrow k+1\end{array}\right)$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$\square$

Proof of (2)

$$E[X^2] = \sum_{k=\cancel{0}\,1}^{\infty} k^2 e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} (k+1) \frac{\lambda^k}{k!} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \left(k \frac{\lambda^k}{k!} + \frac{\lambda^k}{k!} \right)$$

$$= \lambda \left(\underbrace{e^{-\lambda} \cdot \sum_{k=\cancel{0}\,1}^{\infty} k \frac{\lambda^k}{k!}}_{(*)} + e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda (\lambda + e^{-\lambda} \cdot e^{\lambda})$$

$$= \lambda(\lambda+1) = \lambda^2 + \lambda \qquad \blacksquare$$

<u>EX</u> The Quiz Problem

A game consists of 2 questions

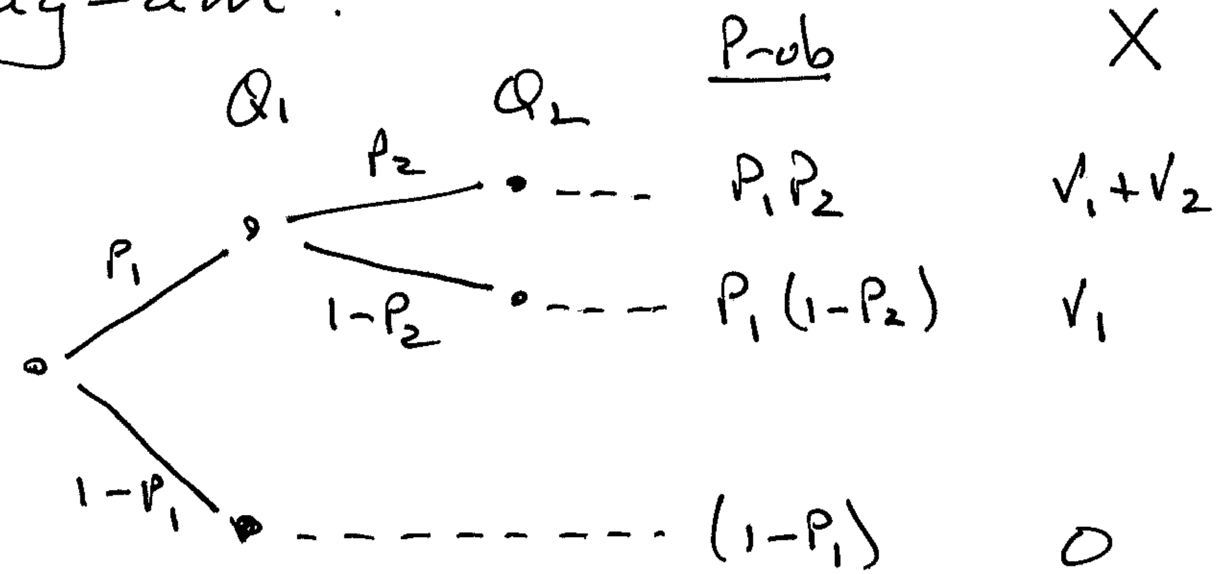
	Q_1	Q_2
P(correct)	p_1	p_2
Payoff ($)	v_1	v_2

The 1st <u>wrong</u> answer terminates the game. You can choose either order $\boxed{(Q_1, Q_2)}$ or (Q_2, Q_1)

assume this

Let X = total payoff.
find $E[X]$.

diagram:



so PMF is:

$$p_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1 + v_2 \\ p_1(1-p_2) & \text{if } x = v_1 \\ 1-p_1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

≥ 0

$$E[X] = (v_1+v_2)p_1p_2 + v_1 p_1(1-p_2)$$

$$= \cancel{v_1 p_1 p_2} + v_2 p_1 p_2 + v_1 p_1 - \cancel{v_1 p_1 p_2}$$

$$= \boxed{v_1 p_1 + v_2 p_1 p_2}$$

Let Y = Payoff if order is (Q_2, Q_1)

⋮

$$E[Y] = \boxed{v_2 p_2 + v_1 p_1 p_2}$$

When do you choose (Q_1, Q_2) over (Q_2, Q_1)?

when

$$E[X] > E[Y]$$

$$V_1P_1 + V_2P_1P_2 > V_2P_2 + V_1P_1P_2$$

$$V_1P_1 - V_1P_1P_2 > V_2P_2 - V_2P_1P_2$$

$$V_1P_1(1-P_2) > V_2P_2(1-P_1)$$

$$\therefore \quad \frac{V_1P_1}{1-P_1} > \frac{V_2P_2}{1-P_2}$$

<u>Exercise</u>: do this for 3, 4, ..., n questions, (answer: see discussion Problems 2.5 #28)

2.5 Joint PMFs of multiple r.v.s

Defn

Given 2 r.v.s X, Y their joint PMF is

$$p_{X,Y}(x,y) = P(X=x, Y=y)$$
$$= P(\{X=x\} \cap \{Y=y\})$$

note! if $R \subseteq \text{range}(X) \times \text{range}(Y)$

Then

$$P((x,y) \in R) = \sum_{(x,y) \in R} p_{X,Y}(x,y)$$

Ex.

Suppose

$$\text{range}(X) = \{1, 2, 3, 4\}$$

$$\text{range}(Y) = \{1, 2, 3\}$$

Suppose joint PMF is

		$x = 1$	$x = 2$	$x = 3$	$x = 4$	$p_Y(y)$
y	3	$1/20$	$1/20$	$2/20$	$1/20$	$5/20$
	2	$2/20$	$3/20$	$3/20$	$1/20$	$9/20$
	1	$1/20$	$1/20$	$1/20$	$3/20$	$6/20$
$p_X(x)$		$4/20$	$5/20$	$6/20$	$5/20$	

$p_{X,Y}(x,y)$

compute $P(Y=3)$. note

$$\{X=1\}\cup\{X=2\}\cup\{X=3\}\cup\{X=4\}=\Omega$$

form a Partition of Ω. By the total Prob. thm

$$P(Y=3)=P(X=1,Y=3)+P(X=2,Y=3)$$
$$+P(X=3,Y=3)+P(X=4,Y=3)$$
$$=\sum_{x=1}^{4} P_{X,Y}(x,3)$$

Marginal PMFs

$$\boxed{P_X(x) = \sum_y P_{X,Y}(x,y)}$$

$$\boxed{P_Y(y) = \sum_x P_{X,Y}(x,y)}$$

Functions of 2 r.v.s

Let $Z = g(X, Y)$ where

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Then

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x,y)$$

Expected Value rule

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)}$$

<u>special case</u>

$$g(x,y) = ax + by + c$$

<u>exercise</u>:

show

$$E[aX + bY + c]$$

$$= aE[X] + bE[Y] + c$$