CSE 107 1-20-26

2.4 Expectation, mean, variance

Defn

The expected value of a r.v. X

is

$$E[X] = \sum_{\substack{x \\ x \in range(X)}} x \cdot \underbrace{p_X(x)}_{P(X=x)}$$

also called the expectation, or mean,

also average.

Ex. (Previous)

$$p_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = (-1)(.2) + 1 \cdot (.3) + 2 \cdot (.5)$$

$$= \boxed{1.1}$$

If $Y = X^2$, then

$$p_Y(y) = \begin{cases} .5 & \text{if } y = 1 \\ .5 & \text{if } y = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X^2] = 1 \cdot (.5) + 4 \cdot (.5) = \boxed{2.5}$$

Expected Value rule

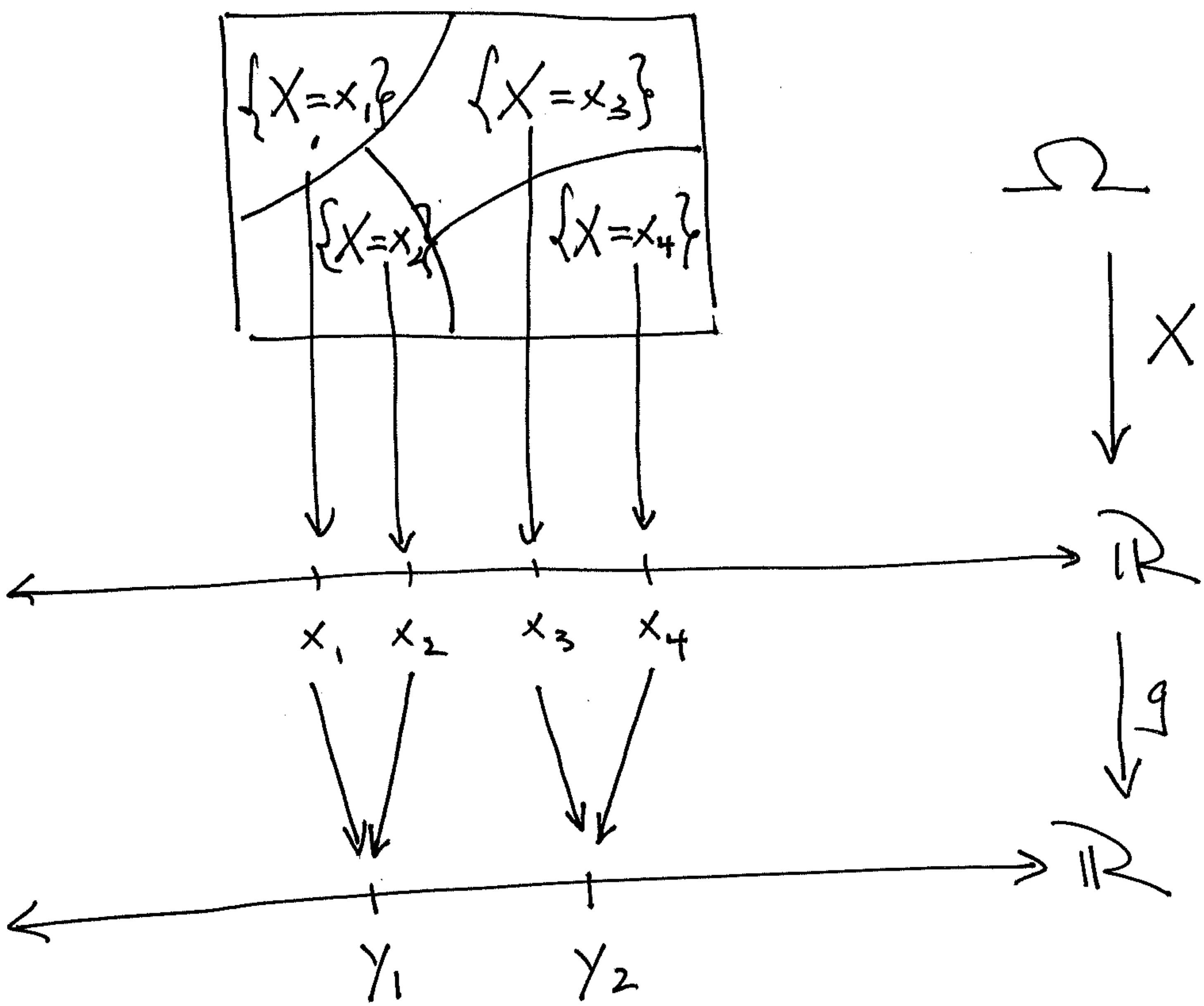
$$\boxed{E[g(X)] = \sum_{x} g(x) \cdot p_X(x)}$$

Ex. (Previous)

$$E[X^2] = (-1)^2 \cdot (.2) + 1^2 \cdot (.3) + 2^2 \cdot (.5)$$

$$= \boxed{2.5}$$

Picture



$$E[Y] = y_1 P_Y(y_1) + y_2 \cdot P_Y(y_2)$$

$$= y_1\left(p_X(x_1)+p_X(x_2)\right) + y_2\left(p_X(x_3)+p_X(x_4)\right)$$

$$= y_1 p_X(x_1)+y_1 p_X(x_2) + y_2 p_X(x_3) + y_2 p_X(x_4)$$

$$= g(x_1)p_X(x_1) + g(x_2)p_X(x_2) + g(x_3)p_X(x_3) + g(x_4)p_X(x_4)$$

$$= \sum_{x \in range(X)} g(x)\cdot p_X(x)$$

$$\therefore \quad E[g(X)] = \sum_x g(x)\cdot p_X(x)$$

Ex. $g(x) = ax + b$

$$Y = g(X) = aX + b$$

$$E[Y] = E[aX + b]$$

$$= \sum_x (ax + b) p_X(x)$$

$$= \sum_x \left(ax\, p_X(x) + b\, p_X(x) \right)$$

$$= a \sum_x x\, p_X(x) + b \sum_x p_X(x)$$

$$= a\, E[X] + b$$

$$\therefore \quad E[aX + b] = a\, E[X] + b$$

In Particular

- $b=0$: $E[aX] = aE[X]$
- $a=0$: $E[b] = b$
- $a=1$: $E[X+b] = E[X]+b$

__Defn__

The __n^{th} moment__ of X $(n=0,1,2,\ldots)$ is $E[X^n]$

so $E[X]$ is the 1^{st} moment.

Defn

The Variance of X is

$$Var(X) = E[(X - E[X])^2]$$

The standard deviation of X is

$$\sigma_X = \sqrt{Var(X)}$$

note: $Var(X)$ and σ_X are measures of dispersion of 'mass' about $E[X]$.

Ex. same

$$p_X(x) = \begin{cases} .2 & \text{if } x=-1 \\ .3 & x=1 \\ .5 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

Previously $E[X]=1.1$, $E[X^2]=2.5$.

$$Var(X) = E[(X-E[X])^2]$$

$$= E[(X-(1.1))^2]$$

$$= (-1-(1.1))^2(.2)+(1-(1.1))^2(.3)+(2-(1.1))^2(.5)$$

$$= \cdots = \boxed{1.29}$$

$$\sigma_X = \sqrt{1.29} = \boxed{1.1358}$$

Variance in terms of moments

$$Var(X) = E[X^2] - (E[X])^2$$

Proof

Write $\mu = E[X]$. Then

$$Var(X) = E[(X-\mu)^2]$$

$$= \sum_x (x-\mu)^2 \cdot p_X(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\mu \sum_x x p_X(x) + \mu^2 \sum_x p_X(x)$$

$$= E[X^2] - 2\mu \cdot \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

$\blacksquare$

<u>Ex</u>. (same)

$$Var(X) = (2.5) - (1.1)^2$$

$$= \ldots = \boxed{1.29}$$

Recall: $E[aX+b] = aE[X]+b$

what is $Var(aX+b)$?

let $Y = aX+b$, and find

2nd moment

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 \cdot p_X(x)$$

$$= \sum_x (a^2x^2 + 2abx + b^2)\, p_X(x)$$

$$= a^2 \sum_x x^2 p_X(x) + 2ab \sum_x x\, p_X(x) + b^2 \sum_x p_X(x)$$

$$= a^2 E[X^2] + 2ab\,E[X] + b^2$$

so

$$Var(aX+b) = Var(Y)$$

$$= E[Y^2] - (E[Y])^2$$

$$= a^2 E[X^2] + \cancel{2ab\,E[X]} + \cancel{b^2} - \underbrace{(a E[X] + b)^2}_{a^2 E[X]^2 + \cancel{2ab\,E[X]} + \cancel{b^2}}$$

$$= a^2 \left(E[X^2] - E[X]^2 \right)$$

$= a^2 \cdot Var(X)$

Summary

- $E[aX+b] = aE[X]+b$
- $Var(aX+b) = a^2 Var(X)$

<u>note:</u>

- if $X = \text{const}$, then $Var(X) = 0$
- $Var(X) \geq 0$ for any r.v. X.
- <u>Exercise</u>: if $Var(X) = 0$, then $X = \text{const}$, i.e. $\text{range}(X) = \{u\}$.

Mean & Variance of

- Bernoulli(p)
- Discrete Uniform(a,b)
- Poisson(λ)

Bernoulli(p)

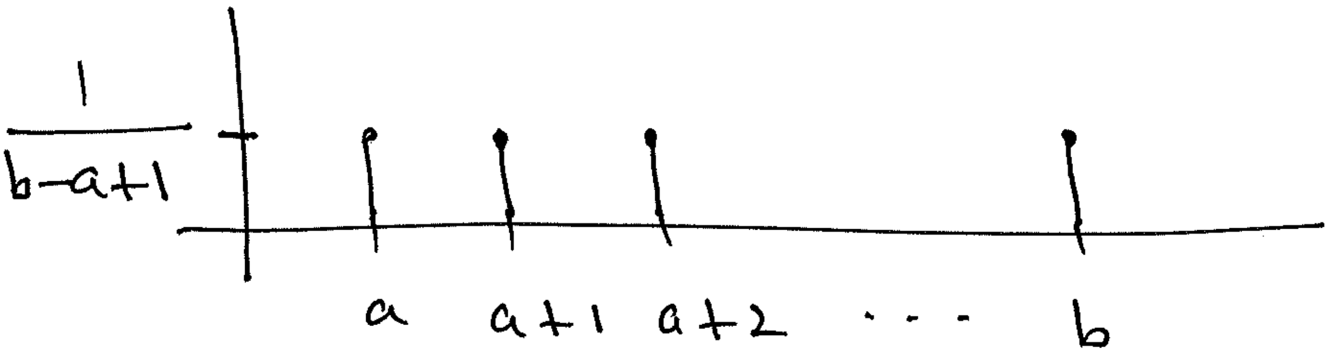
$$P_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$$

$$E[X] = 1\cdot p + 0\cdot(1-p) = \boxed{p}$$

$$E[X^2] = 1^2\cdot p + 0^2\cdot(1-p) = p$$

$$Var(X) = p - p^2 = \boxed{p(1-p)}$$

Discrete Unif(a, b)



$$P_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{k=a}^{b} k \cdot \left(\frac{1}{b-a+1}\right)$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \sum_{k=a}^{b} k$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k\right)$$
