

CS2 107 1-16-26

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Supplemental Lecture

The Binomial r.v.

Perform a sequence of n indep.

Bernoulli trials with parameter p .

Let

$$X(\omega) = \# \text{ of successes}$$

↑

a bit string
of len. n

PMF : Binomial Probabilities

$$P_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k=0, 1, 2, \dots, n$.

more specifically

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k=0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

note

$$\sum_{k=0}^n P_X(k) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (p + (1-p))^n = 1^n = 1 \quad \checkmark$$

note: 2 Parameters: n, p

The Geometric r.v.

Perform indep. Bernoulli trials
(same param. p) until the 1st
success. Let

$$X(\omega) = \text{length of sequence} = \# \text{ trials}$$

↑
bit string of
form $000\dots 01$

$\text{range}(X) = \{1, 2, 3, \dots\}$

what is $p_X(k) = P(X=k)$?

$$p_X(k) = (1-p)^{k-1} \cdot p \quad \text{for } k=1, 2, \dots$$

more precisely

$$p_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k=1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

observe

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$= p \cdot \sum_{k=0}^{\infty} (1-p)^k \quad \left\{ \begin{array}{l} \text{substitute} \\ k \leftarrow k+1 \end{array} \right.$$

$$= p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1 \quad \checkmark$$

note! Geometric has one Parameter,
 p , Prob. of Success.

The Poisson r.v.

a Poisson r.v. has PMF

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \text{if } k=0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

note: did not specify $X: \Omega \rightarrow \mathbb{R}$

absolve

$$\begin{aligned} \sum_{k=0}^{\infty} P_X(k) &= \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \end{aligned}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1 \quad \checkmark$$

Parameter: $\lambda > 0$

note $\text{Poisson}(\lambda)$ approximates
 $\text{Binomial}(n, p)$, where $\lambda = np$
 n is large, p is small.

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k}$$

more precisely: let $\lambda > 0$ be
 constant, $p = \frac{\lambda}{n}$, then

$$\lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Ex.

Let $n=200$, $\lambda=2$, $p=\frac{\lambda}{n}=.01$.

Then the Prob. of $K=4$ successes in 200 indep. Bernoulli(p) is

$$\binom{200}{4} (.01)^4 (.99)^{196} = \underline{.0902197}$$

This is well approximated by

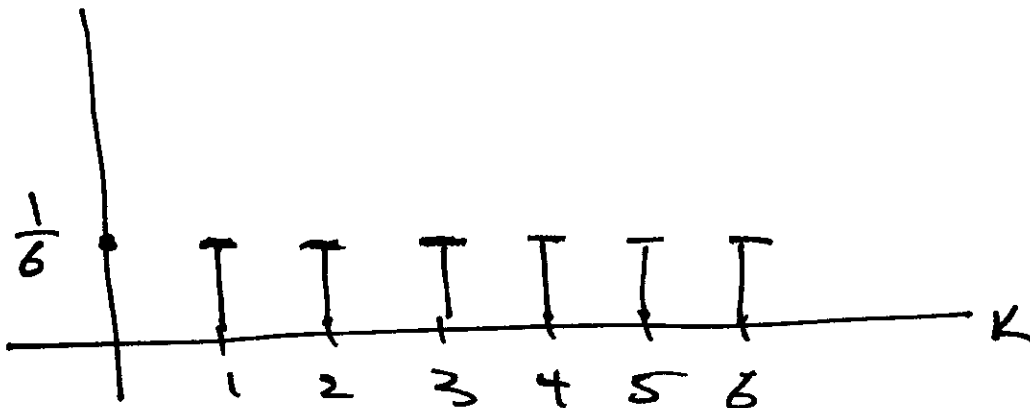
$$e^{-2} \cdot \frac{2^4}{4!} = \underline{.0902235}$$

Discrete Uniform r.v.

Ex. Roll 1 fair die (6-sided). Let X = the number showing on die.
 $\Rightarrow \text{range}(X) = \{1, 2, 3, 4, 5, 6\}$

$$P_X(k) = P(X=k) = \begin{cases} \frac{1}{6} & \text{if } k=1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

graph



In general a discrete uniform
r.v. has $\text{range}(X) = \{a, a+1, a+2, \dots, b\}$

and

$$P_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k=a, a+1, \dots, b \\ 0 & \text{otherwise} \end{cases}$$

note

$$\sum_{k=a}^b \left(\frac{1}{b-a+1} \right) = (b-a+1) \cdot \frac{1}{b-a+1} = 1 \quad \checkmark$$

2.3 Functions of a random variable

Given a r.v. X , we may transform X by composition with

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

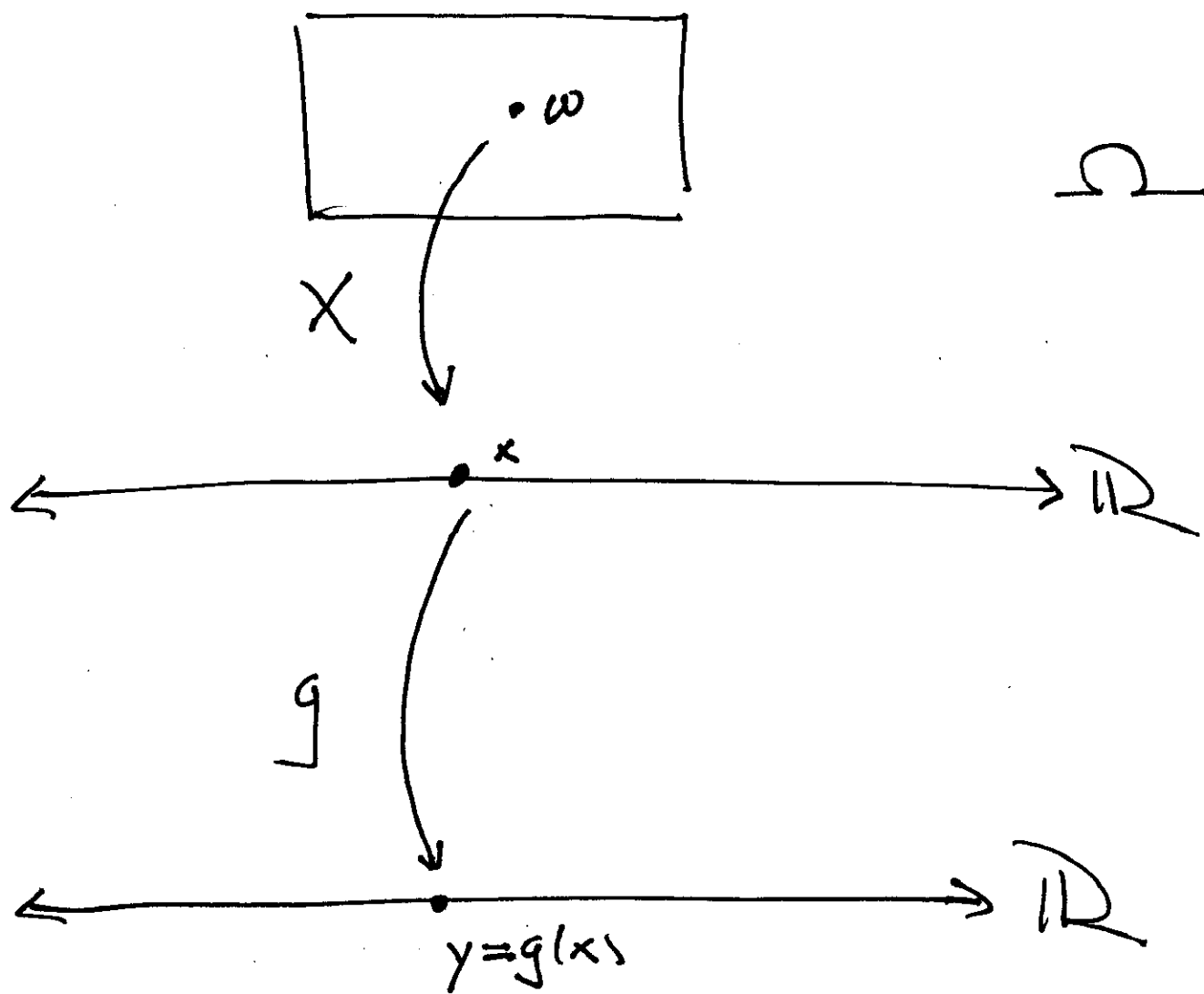
to obtain $Y = g \circ X = g(X)$.

Since

$$X: \Omega \rightarrow \mathbb{R}$$

we have $Y = g \circ X: \Omega \rightarrow \mathbb{R}$,

so Y is itself a new random variable.



Ex.Suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$. Find PMF of Y .
(g(x) = x^2)

$$P_Y(y) = P(Y=y)$$

$$= P(X^2 = y)$$

$$= P(X = \pm\sqrt{y})$$

$$= P(X = \sqrt{y} \text{ or } X = -\sqrt{y})$$

$$= P(X=\sqrt{y}) + P(X=-\sqrt{y})$$

$$= \left\{ \begin{array}{l} .2 \text{ if } \sqrt{y} = -1 \\ .3 \text{ if } \sqrt{y} = 1 \\ .5 \text{ if } \sqrt{y} = 2 \end{array} \right\} + \left\{ \begin{array}{l} .2 \text{ if } -\sqrt{y} = -1 \\ .3 \text{ if } -\sqrt{y} = 1 \\ .5 \text{ if } -\sqrt{y} = 2 \end{array} \right\}$$

$$= \left\{ \begin{array}{l} .3 \text{ if } y=1 \\ .5 \text{ if } y=4 \\ 0 \text{ otherwise} \end{array} \right\} + \left\{ \begin{array}{l} .2 \text{ if } y=1 \\ 0 \text{ otherwise} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} .5 \text{ if } y=1 \\ .5 \text{ if } y=4 \\ 0 \text{ otherwise} \end{array} \right\}$$

