CSE 107 1-15-26

Binomial Probabilities

Defn A Bernoulli Trial is an experiment with 2 possible outcomes

success = head = 1
failure = tails = 0

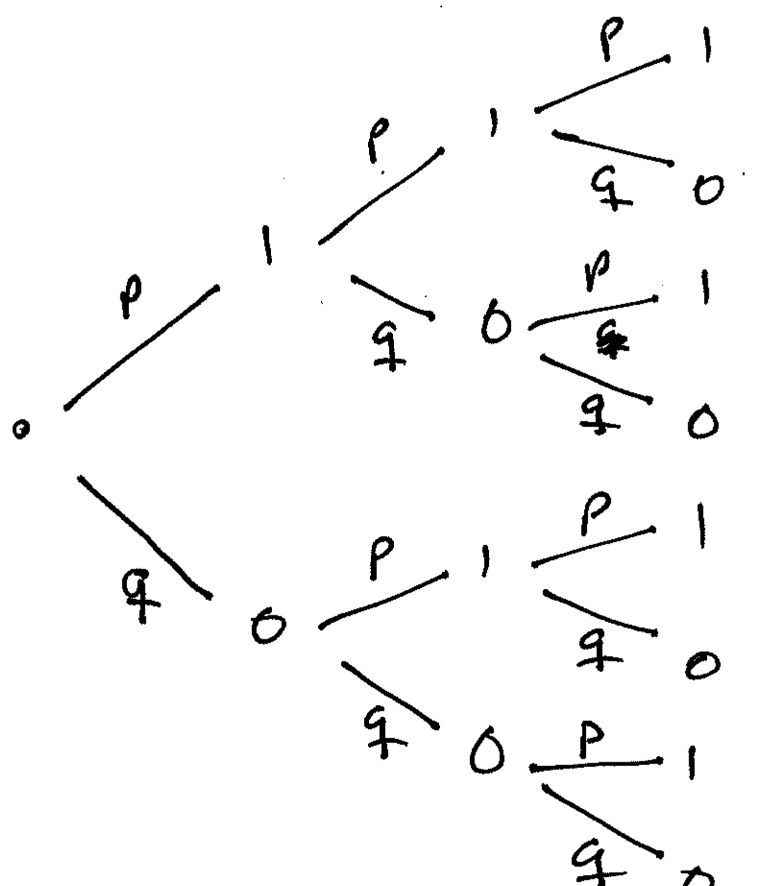
<u>Ex.</u>

Perform a seq. of 3 indep. tosses of a (weighted) coin.

Say

$$P(H) = p \, , \; P(T) = q$$

<u>note</u>: $p + q = 1 \quad \therefore \quad q = 1 - p$



outcomes	Prob.
111	p^3
110	$p^2 q$
101	$p^2 q$
100	$\boxed{p q^2}$
011	$p^2 q$
010	$\boxed{p q^2}$
001	$\boxed{p q^2}$
000	q^3

note

$$p^3 + 3p^2q + 3pq^2 + q^3 = (p+q)^3 = 1$$

$$P(3 \text{ heads}) = P(111) = p^3$$

$$P(2 \text{ heads}) = P(\{110, 101, 011\}) = 3 \cdot p^2 \cdot q^1$$

$$P(1 \text{ head}) = P(\{100, 010, 001\}) = 3pq^2$$

$$P(0 \text{ heads}) = P(000) = q^3$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

here: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

also

$$\binom{n}{k} = \boxed{\begin{array}{l}\#\ k\text{-subsets} \\ \text{of an } n\text{-set}\end{array}} = \boxed{\begin{array}{l}\#\text{ Bit strings of} \\ \text{len. } n \text{ with} \\ \text{exactly } k \text{ 1's}\end{array}}$$

<u>Thm</u>

If we perform n indep. Bernoulli trials, then

$$\underbrace{P(\#\text{successes} = k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}}$$

Binomial Probabilities

1.6 Combinatorics

Read 1.6

- # of Permutations of n objects is : $n!$

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

- # of k-Permutations of n objects

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- # of k-Combinations of n objects

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \boxed{} = \boxed{}$$

$0 \le k \le n$

- # partitions of n objects into k sets with set i having n_i objects $(n_1+n_2+\cdots+n_k=n)$

<u>multinomial coefficient</u>

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

- k-combinations of n objects where repitition is allowed.

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

Ex. $n=4$, $k=2$, $S=\{1,2,3,4\}$

$S = \{1, 2, 3, 4\}$

Bitstrs		Subsets
0 0 0 0	$\longleftrightarrow$	$\emptyset$
0 0 0 1	$\longleftrightarrow$	$\{4\}$
0 0 1 0	$\longleftrightarrow$	$\{3\}$
$\boxed{0\ 0\ 1\ 1}$	$\longleftrightarrow$	$\boxed{\{3,4\}}$
0 1 0 0	$\longleftrightarrow$	$\{2\}$
$\boxed{0\ 1\ 0\ 1}$	$\longleftrightarrow$	$\boxed{\{2,4\}}$
$\boxed{0\ 1\ 1\ 0}$	$\longleftrightarrow$	$\boxed{\{2,3\}}$
0 1 1 1	$\longleftrightarrow$	$\{2,3,4\}$
1 0 0 0	$\longleftrightarrow$	$\{1\}$
$\boxed{1\ 0\ 0\ 1}$	$\longleftrightarrow$	$\boxed{\{1,4\}}$
$\boxed{1\ 0\ 1\ 0}$	$\longleftrightarrow$	$\boxed{\{1,3\}}$
1 0 1 1	$\longleftrightarrow$	$\{1,3,4\}$
$\boxed{1\ 1\ 0\ 0}$	$\longleftrightarrow$	$\boxed{\{1,2\}}$
1 1 0 1	$\longleftrightarrow$	$\{1,2,4\}$
1 1 1 0	$\longleftrightarrow$	$\{1,2,3\}$
1 1 1 1	$\longleftrightarrow$	$\{1,2,3,4\}$

note

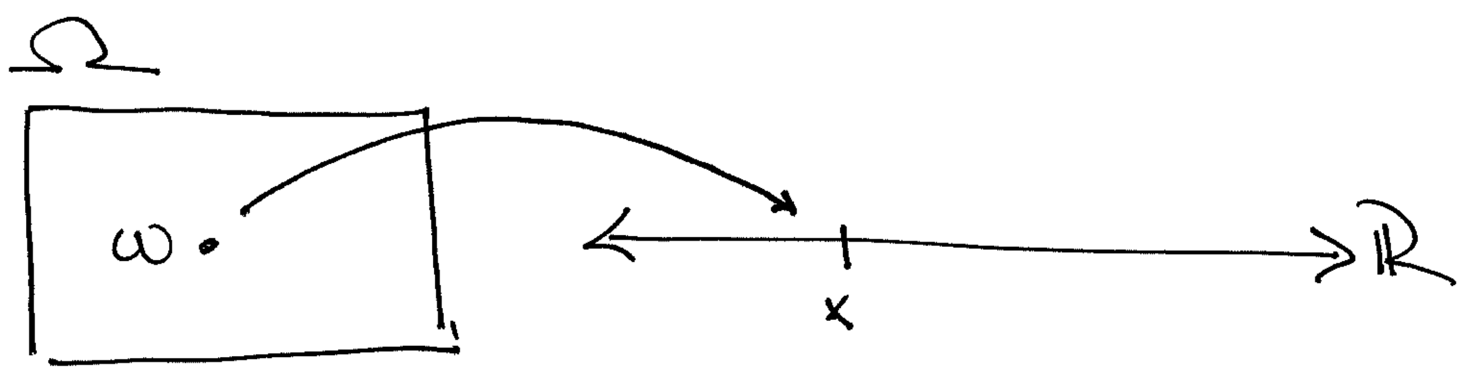
$\binom{4}{2} = \frac{\cancel{4}^{2}\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot\cancel{2}} = 6$

2.1 Random Variables

Defn

A <u>random variable</u> is a function

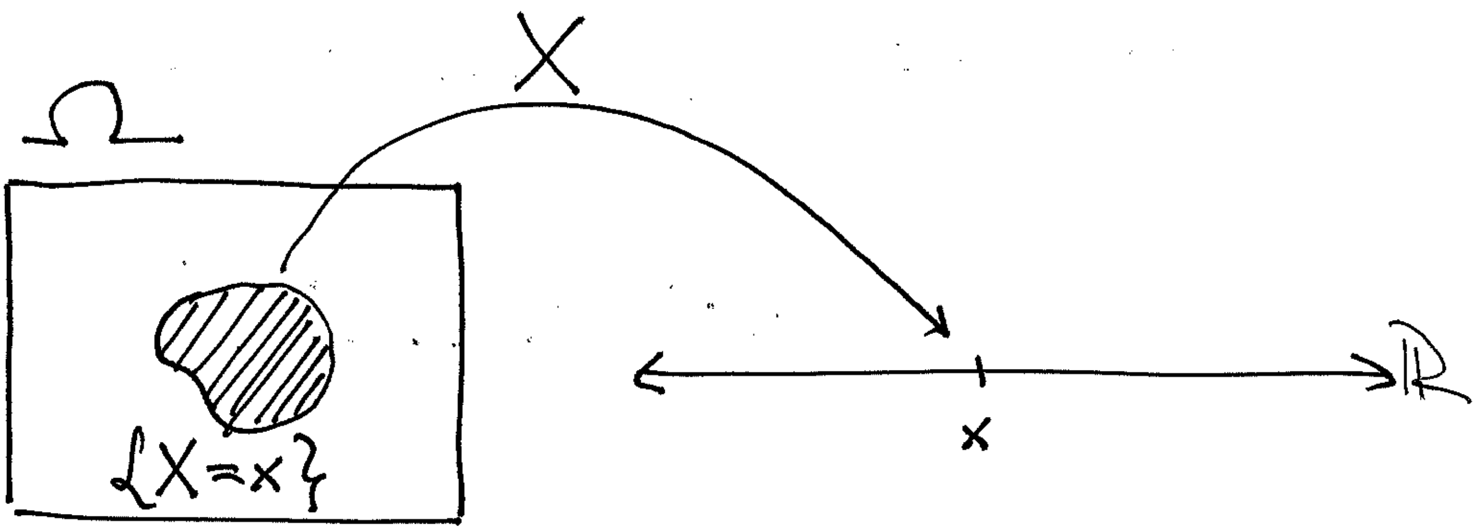
$$X : \Omega \longrightarrow \mathbb{R}$$



Notation $\quad X(\omega) = x$

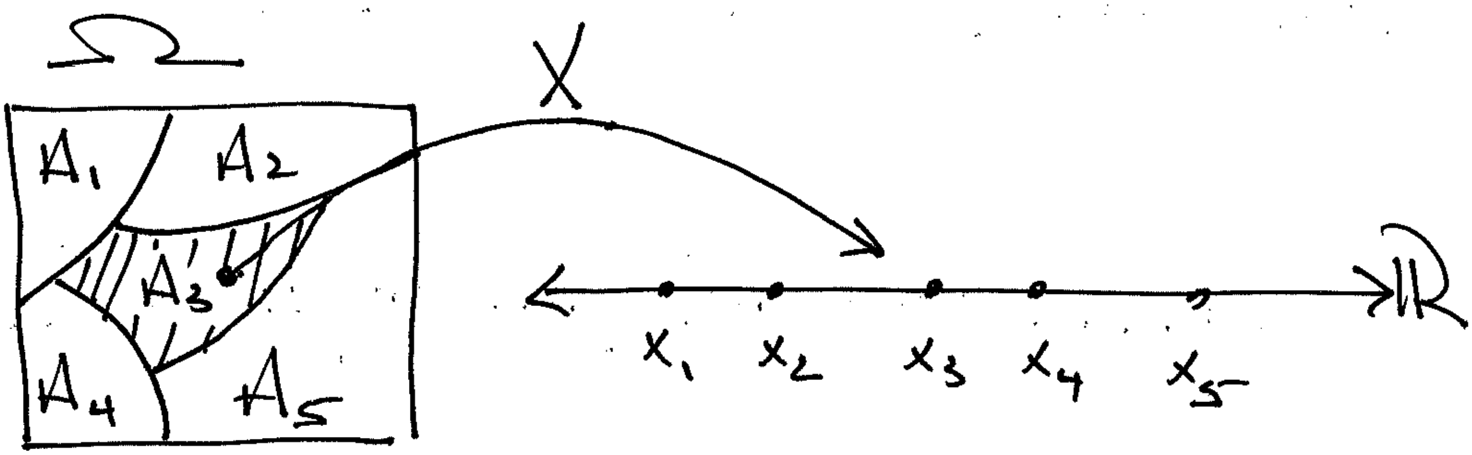
- We denote by $\{X=x\}$ the event

$$\{X=x\}=\{\omega\in\Omega \mid X(\omega)=x\}\subseteq\Omega$$



- X is called <u>discrete</u> iff its range is a discrete subset of $\mathbb{R}$.

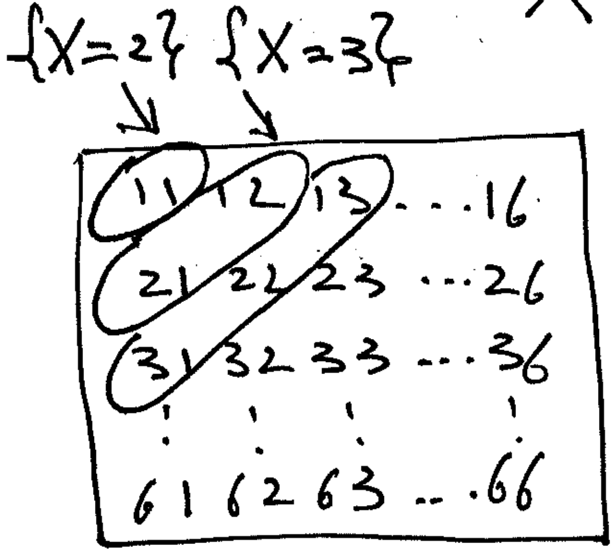
$$\text{range}(X)=\{X(\omega) \mid \omega\in\Omega\}\subseteq\mathbb{R}$$



$$A_i = \{X = x_i\}$$

Ex. Roll 2 fair indep. dice let

$$X(ij) = i + j$$

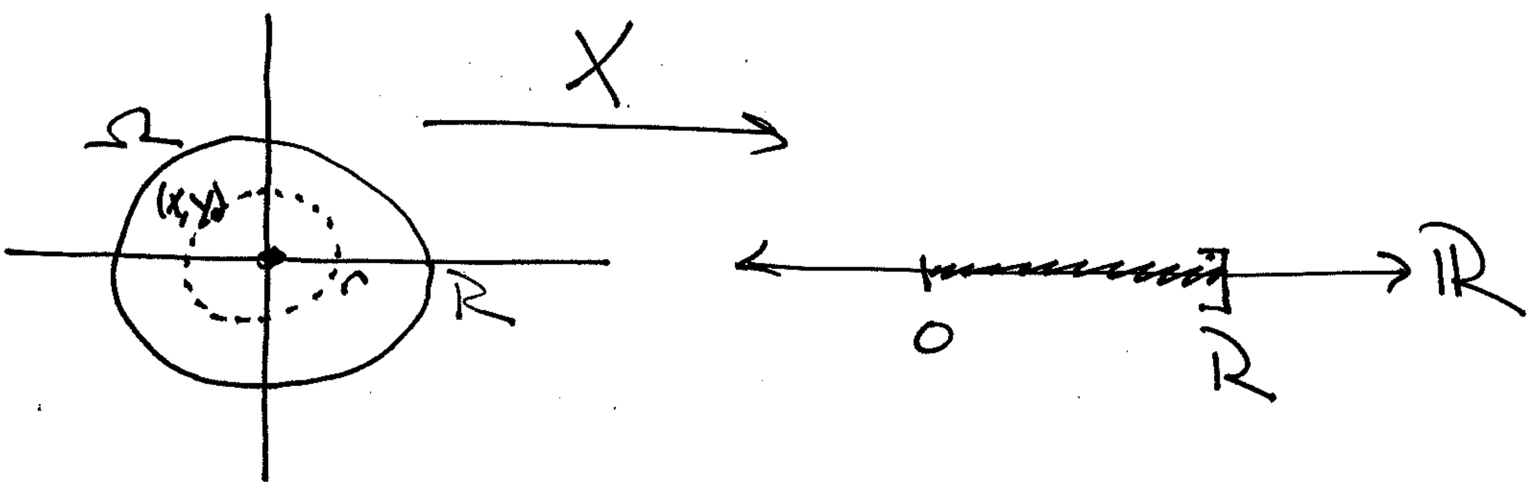


range(X) = $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$$\Omega \xrightarrow[X]{} \mathbb{R}$$

Ex. continuous r.v.

throw a dart at a circular target centered at (0,0), of radius R.



$$\mathrm{range}(X) = [0, R]$$

$$X((x,y)) = \sqrt{x^2+y^2}$$

$$= \text{distance } (x,y) \text{ to } (0,0)$$

2.2 Probability Mass Functions

<u>Defn</u>

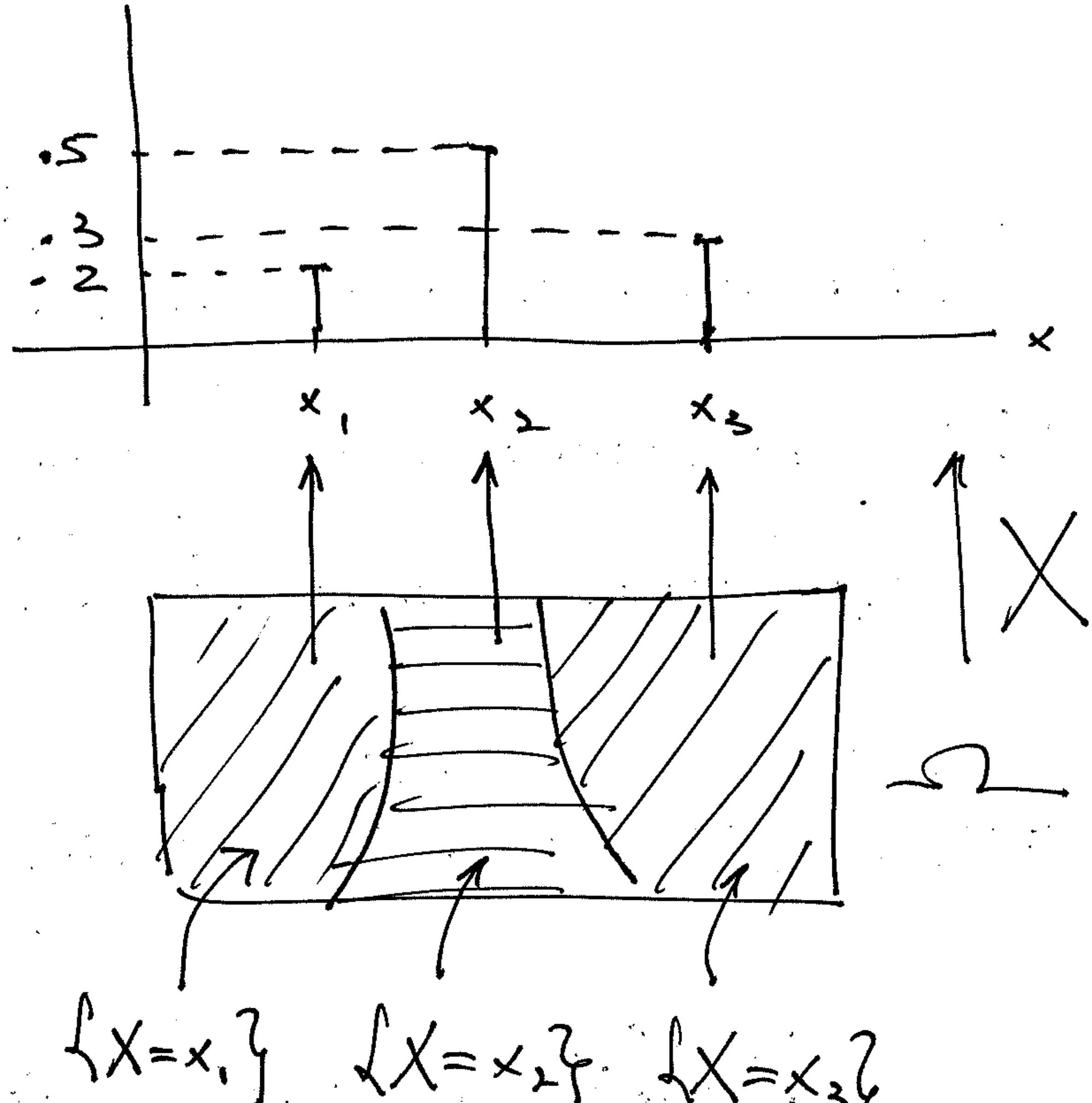
Let X be a discrete r.v. The Probability Mass Function (PMF) of X is the function

$$p_X(x) = P(\{X = x\}) = P(X = x)$$

observe: $p_X : \mathbb{R} \longrightarrow [0, 1]$

also: $\displaystyle\sum_{x \in \text{range}(X)} p_X(x) = 1$

Ex.



so

$$P_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

The Bernoulli Rand. Var.

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is success} \\ 0 & \text{if } \omega \text{ is failure} \end{cases}$$

its PMF is

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$