CSE 107
Final Exam Review Problems

1. You are in a class with 200 other students. Let X be the number of other students in the class who share your birthday. For simplicity, exclude birthdays on February 29, and assume all birthdays are equally likely.

   a. Find the probability that *exactly one* other student in the class shares your birthday.

   b. Find the probability that *at least one* other student in the class shares your birthday.

2. We perform independent rolls of a fair 6-sided die until the number on top is at least 5. Let the random variable X be the number of rolls performed.

   a. Determine the PMF of X.

   b. Determine $P(X \leq 3)$.

3. Let X and Y be independent exponential random variables, both with parameter λ. Use the convolution product to find the PDF $f_Z(z)$ of the random variable $Z = X + Y$.

4. Let X be uniform on $[0, 2]$, Y be uniform on $[1, 4]$, and let $Z = X + Y$.

   a. Determine the PDF of the random variable Z.

   b. Determine $P(4 \leq Z \leq 6)$.

5. Let X be a continuous random variable with PDF $f_X(x)$, and let Y be a discrete random variable with PMF

$$p_Y(y) = \begin{cases} 1/3 & \text{if } y = 1 \\ 2/3 & \text{if } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose X and Y are independent, and let $Z = XY$.

   a. Determine the CDF $F_Z(z)$ in terms of the CDF of X.

   b. Use your answer in part (a) to deduce that Z is a continuous random variable, and find its PDF $f_Z(z)$ in terms of the PDF of X.

6. Let X be a discrete random variable with $\text{range}(X) = \{1, 2, 3\}$ (i.e. think of X as a 3-sided die.) Let A denote a Bernoulli process with parameter p (i.e. think of A as a bit stream in which 1 is considered an "arrival"). Assume the process A is independent of X. Suppose further that the stream A is split into three streams B_1, B_2 and B_3 according to the following rule. Whenever an arrival occurs in A, we sample X (i.e. throw the die) and if $X = i$, then an arrival occurs in B_i, for $i = 1, 2, 3$. You may assume that the streams B_1, B_2 and B_3 are themselves Bernoulli processes.

   a. Let $p_X(1) = q$, $p_X(2) = r$ and $p_X(3) = s$. Determine the parameters of the processes B_1, B_2 and B_3 in terms of p, q, r and s.

   b. Suppose B_1, B_2 and B_3 have parameters 0.1, 0.1 and 0.2 respectively. Determine p.

   c. Use your answers to parts (a) and (b) to determine the PMF $p_X(x)$.

7. Let $\mathcal{A}$ be a Bernoulli process in which the probability of an arrival is $p = 0.8$. We split $\mathcal{A}$ into two new Bernoulli processes $\mathcal{B}$ and $\mathcal{C}$ by flipping a coin with $P(\text{Head}) = q = 0.4$. Upon each arrival in $\mathcal{A}$, we flip the coin. If the coin is heads, the arrival goes to $\mathcal{B}$, if tails, the arrival goes to $\mathcal{C}$.

   a. What is the expected time of the 12^{th} arrival in $\mathcal{B}$?

   b. What is the mean of the interarrival times in $\mathcal{C}$?

8. During morning rush hour, accidents on a certain 40 mile stretch of Highway 101 occur as a Poisson process with rate $\lambda = 1.5/\text{hour}$. Assume that the accidents occurring in the northbound lanes are independent of those occurring in the southbound lanes, and that these two groups of accidents themselves comprise two Poisson processes with arrival rates λ_N and λ_S, respectively. (We assume that no accident occurs on *both* sides.)

   a. Suppose $\lambda_N = 1.2/\text{hour}$. Determine λ_S.

   b. For any particular accident occurring on this stretch of Highway101, what is the probability that it occurs in the southbound lanes.

   c. What is the expected time from the beginning of rush hour to the 3^{rd} northbound accident.

9. Customers enter a department store through two entrances, East and West. Arrivals at these entrances constitute two independent Poisson processes. The East entrance has an arrival rate of $\lambda_1 = 0.3$ persons per minute, while the West entrance has an arrival rate of $\lambda_2 = 0.5$ persons per minute.

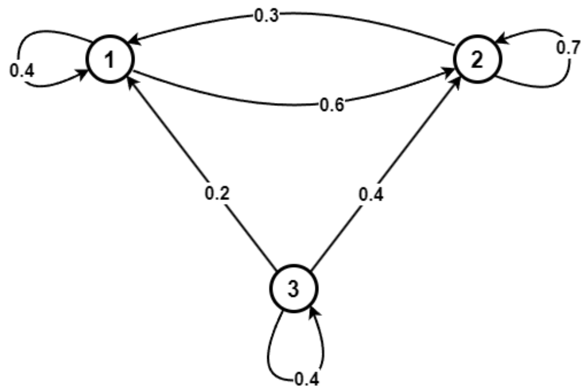
   a. Find the probability that 6 persons enter the store (from any direction) within a 10 minute period.

   b. Find the expected time until the 10^{th} customer arrives.

   c. If a customer walks into the store, what is the probability that it is through the East entrance?

   d. Suppose you stand at the East entrance. What is your expected wait time until the next person walks through the door?

10. Consider a 3-state Markov chain model with the following state transition diagram.



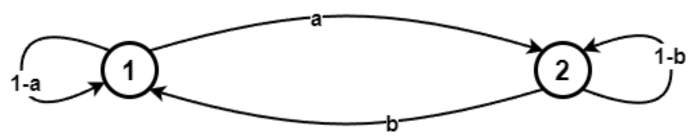
a. Write the state transition matrix R.

b. Find $P(X_2 = 2 \mid X_0 = 3)$.

c. Find $P(X_3 = 2 \mid X_1 = 3)$.

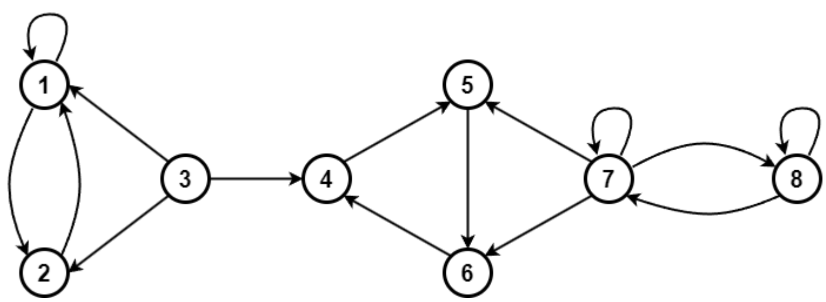
d. Find $P(X_{101} = 2 \mid X_{100} = 1)$.

11. Let $0 < a < 1$ and $0 < b < 1$ and consider the 2-state Markov chain model with the following state transition diagram.



Determine the steady-state probabilities of this Markov process.

12. Consider the Markov chain model with the following state transition diagram.



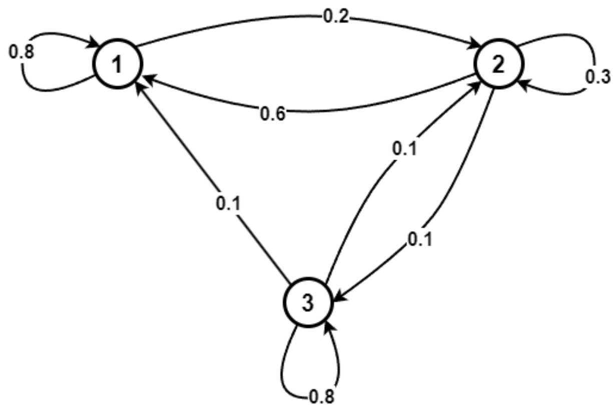
a. Determine $A(i)$ for each state $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

b. Determine which states are recurrent and which are transient.

c. Determine any periodic recurrent classes, and state the period.

d. Determine any aperiodic recurrent classes.

13. Consider the Markov chain model with the following state transition diagram.



Determine the steady-state probabilities of this Markov process.

14. Consider a Markov chain with 2 states $S = \{1, 2\}$, and transition probabilities: $p_{11} = 3/4$, $p_{12} = 1/4$, $p_{21} = 2/5$ and $p_{22} = 3/5$. (Hint: before continuing, draw the state transition diagram.)

a. Find $P(X_2 = 2 \mid X_0 = 2)$.

b. Determine the steady-state probabilities.