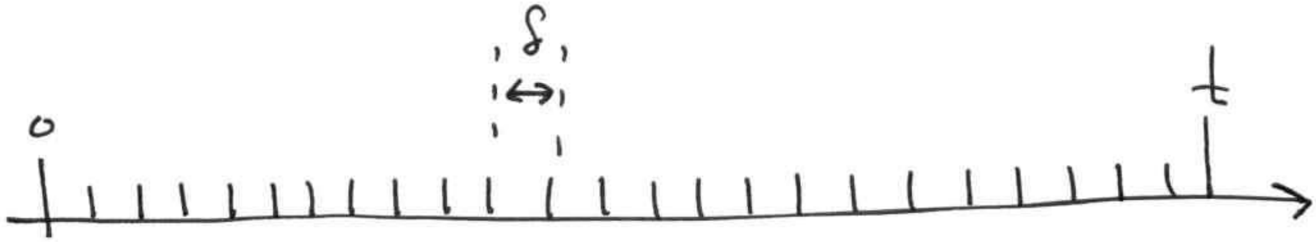
CSE 107 3-8-25

Supplemental Lecture

The Poisson Process

⋮

<u>large time intervals</u>



let $n = \frac{t}{\delta}$ be the number of sub-intervals. so $\delta = \frac{t}{n}$. we choose δ small enough (or n large enough) so that the prob. of more than one arrival in any subinterval is negligable.

we now have, approximately, a Bernoulli Process with Probability of arrival

$$p = \lambda \delta = \frac{\lambda t}{n}$$

The approx. improves as $\delta \to 0$ $(n \to \infty)$.

The # of successes in n trials of a Bernoulli Process is Binomial with Parameters n, p. So, approximately

$P(k \text{ arrivals in } [0, t])$

$$\approx \binom{n}{k} p^k (1-p)^{n-k} \qquad (k = 0, 1, 2, \ldots)$$

$$= \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^k$$

In the limit as $n \to \infty$ $(\delta \to 0)$ This becomes the Poisson PMF with Parameter λt

Thus

$$P(k \text{ arrivals in } [0,t]) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

Let N_t denote the # of arrivals of Poisson Process in $[0,t]$. Then

$$\boxed{P_{N_t}(k) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}}$$

Thus

$$E[N_t] = \lambda t$$

and

$$Var(N_t) = \lambda t$$

Let T be the time to the 1^{st} arrival. Observe

$$\{T > t\} = \{N_t = 0\}$$

and

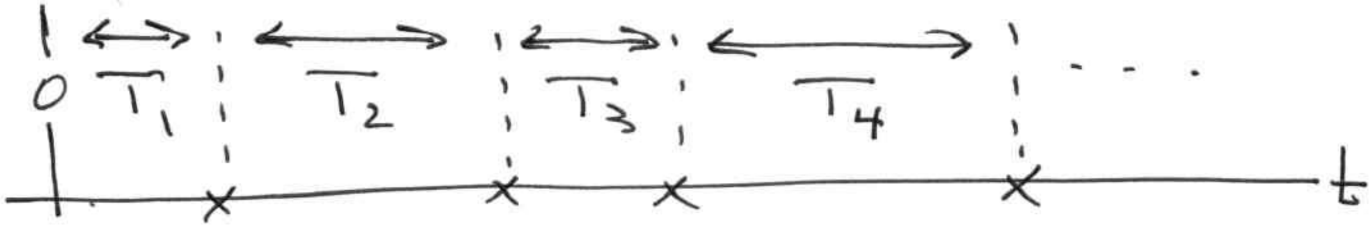
$$\begin{aligned} F_T(t) &= P(T \le t) \\ &= 1 - P(T > t) \\ &= 1 - P(N_t = 0) \\ &= 1 - p_{N_t}(0) \\ &= 1 - e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

we differentiate w.r.t. t to get

$$f_T(t) = \lambda e^{-\lambda t} \quad (\text{for } t \geq 0)$$

Thus T is exponential with param. λ.

Now let T_k be the k^{th} interarrival time $(k = 1, 2 \ldots)$



Recall that the exponential r.v. is memoryless (see notes 2/14/25)

$\therefore$ the Poisson Process has a fresh start after each arrival.

so each T_k is exponential with param. λ, and has PDF

$$f_{T_k}(t) = \lambda e^{-\lambda t} \qquad (t \geq 0)$$

Now define

$$Y_k = T_1 + T_2 + \cdots + T_k \qquad (k = 1, 2 \ldots)$$

i.e. Y_k is the time of the k^{th} arrival. The PDF of Y_k is

$$\boxed{f_{Y_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \qquad (t \geq 0)}$$

Y_k is called the Erlang r.v. of order k (see text.)

Exercise

Show that the sum of k indep. exponential r.v.s (Param λ) is Erlang of order k (Param λ)

Hint use continuous convolution Product and induction on k. Note when $k=1$, Erlang is exponential.

Use convolution on induction step to show

$$Y_k = Y_{k-1} + T_k$$

is Erlang of order k.

Summarize and compare Bernoulli and Poisson Processes.

	Bernoulli	Poisson
time:	$t \in \{1, 2, \dots\}$	$t \in [0, \infty)$
rate:	p/trial	λ/unit time
N_t:	Binomial(t, p)	Poisson(λt)
T_k:	Geometric(p)	Exponential(λ)
Y_k:	Pascal(p, k)	Erlang(λ, k)

Ex.

You receive email according to a Poisson Process at rate of

$$\lambda = 2 \text{ messages/hour}$$

You currently have no new messages

a.) what is the prob. that you find 3 new messages when you check 30 minutes from now ?

$$P(N_{\frac{1}{2}} = 3) = P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-(2\cdot\frac{1}{2})} \cdot \frac{(2\cdot\frac{1}{2})^3}{3!}$$

$$= e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) what is the prob. of no messages in 30 minutes?

$$P(N_{\frac{1}{2}} = 0) = P_{N_{\frac{1}{2}}}(0)$$

$$= e^{-(2 \cdot \frac{1}{2})} \cdot \frac{(2 \cdot \frac{1}{2})^0}{0!}$$

$$= e^{-1} = \boxed{.3679}$$

c.) what is the expected time of your 5^{th} new message?

$$E[Y_5] = E[T_1 + \cdots + T_5]$$

$$= 5 \cdot E[T]$$

$$= 5 \cdot \frac{1}{\lambda} = \frac{5}{2} = \boxed{2.5 \text{ hours}}$$

The sum of Poisson r.v.s

Let X, Y be indep. Poisson r.v.s with parameters λ_1, λ_2 respectively.

Then

$$Z = X + Y$$

is Poisson with parameter $\lambda_1 + \lambda_2$.

Proof

we have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \quad (k=0,1,\dots)$$

and

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \quad (k=0,1,\dots)$$

Thus

$$P_Z(n) = \sum_{k=0}^{\infty} P_X(k) \cdot P_Y(n-k)$$

$$= \sum_{k=0}^{\infty} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{(n-k)}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} \cdot \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{\infty} \binom{n}{k} \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot (\lambda_1 + \lambda_2)^n$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}$$

$\therefore$ Z is Poisson with Param. $\lambda_1 + \lambda_2$. ■

Ex (6.2 #8)

During rush hour, 8–9 am, accidents occur with rate 5/hour, as a Poisson Process. During 9–11 am accidents occur as a Poisson Process with rate 3/hour

what is the PMF of the total # of accidents from 8–11am?

solution

let

N_1 = # accidents in 8–9 am

N_2 = " " " 9–11 am

and

$N = N_1 + N_2$ = # accidents in 8–11 am.

Then

N_1 is Poisson with $\lambda_1 = 5 \cdot 1 = 5$.

N_2 is Poisson with $\lambda_2 = 3 \cdot 2 = 6$.

Thus N is Poisson with Parameter $\lambda = \lambda_1 + \lambda_2 = 11$. Hence

$$P_N(n) = e^{-11} \cdot \frac{11^n}{n!}.$$