CSE 107 3-7-25

Merging two Bernoulli Processes

Suppose we have 2 independent Bernoulli Processes

	Parameter
S1 : $X_1, X_2, \ldots\ldots$	p
S2 : $Y_1, Y_2, \ldots\ldots$	q

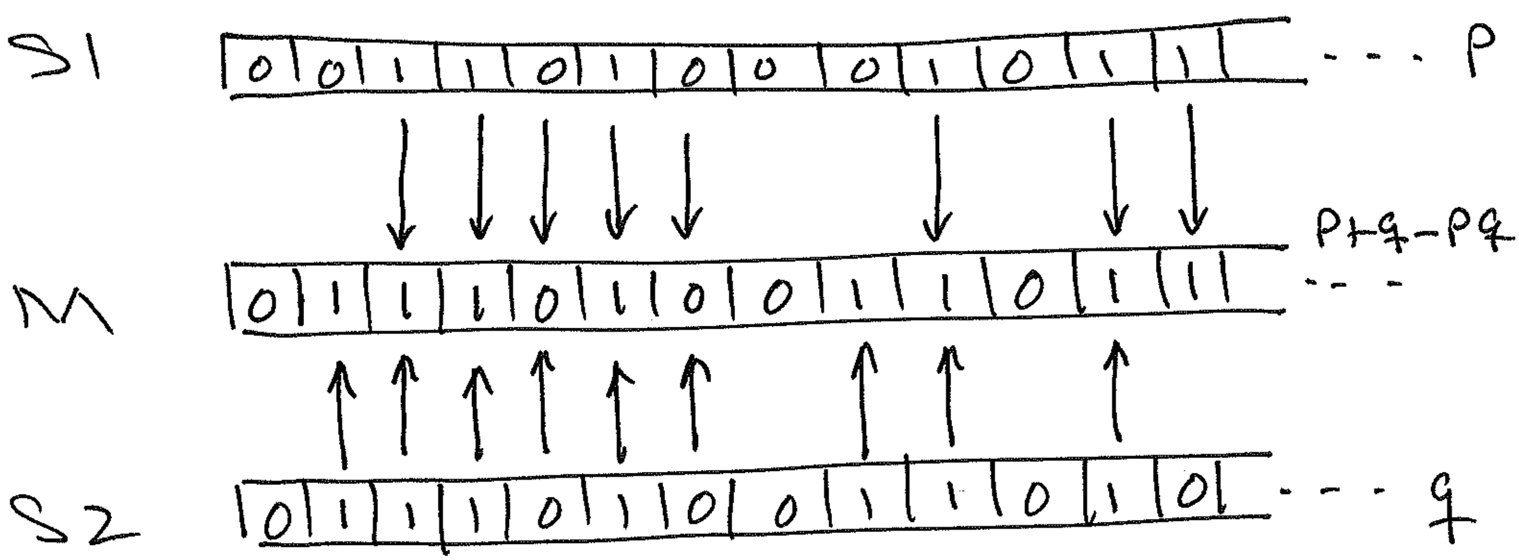
We merge S1 and S2 into M

M : $M_1, M_2, \ldots\ldots$

Rule

$$M_t = \begin{cases} 1 & \text{if either } X_t = 1 \text{ or } Y_t = 1 \\ 0 & \text{otherwise} \end{cases}$$

what kind of Process is M?



observe

$$P(M_t=1) = 1 - P(M_t=0)$$

$$= 1 - P(X_t=0, Y_t=0)$$

$$= 1 - P(X_t=0) \cdot P(Y_t=0)$$

$$= 1 - (1-p)(1-q)$$

$$= 1 - (1 - p - q + pq)$$

$$= p + q - pq$$

Is M a Bernoulli Process ?

ans. Yes !

Exercise

- first split, then merge, find the parameter of resulting B.P.s
- first merge, then split, find the parameters of the B.P.s
- change bit operation to *and*, *exor*, *nand*, *nor*, find the parameters of resulting B.P.s.

6.2 The Poisson Process

Recall an infinite sequence of r.v.'s

$$\{X_t \mid t = 1, 2, 3, \dots\}$$

is a Bernoulli Process iff

(1) each X_t is Bernoulli with same param. p.

(2) the set $\{X_t \mid t = 1, 2, \dots\}$ is independent.

We defined

$$Y_K = \text{time of } K^{th} \text{ arrival/success}$$

we found

$$P_{Y_K}(t) = \binom{t-1}{K-1} \cdot p^K (1-p)^{t-K} \quad \begin{cases} \text{for} \\ t = K, K+1, \ldots \end{cases}$$

Also the interarrival times are

$$\begin{cases} T_1 = Y_1 \\ T_K = Y_K - Y_{K-1} \qquad K = 2, 3, \ldots \end{cases}$$

can also define

$$N_t = X_1 + X_2 + \dots + X_t \quad (t=1,2,\dots)$$

the # of arrivals/successes up to time t. The PMF of N_t is

$$P_{N_t}(k) = \binom{t}{k} p^k (1-p)^{t-k} \quad \begin{cases} \text{for} \\ k=0,1,2\dots \end{cases}$$

Note: we can recover X_t from N_t

$$\begin{cases} X_1 = N_1 \\ X_t = N_t - N_{t-1} \quad t=2,3\dots \end{cases}$$

Also, can recover Y_k from N_t

$$Y_k = \min\{t \mid N_t = k\} \qquad k=1,2,\ldots$$

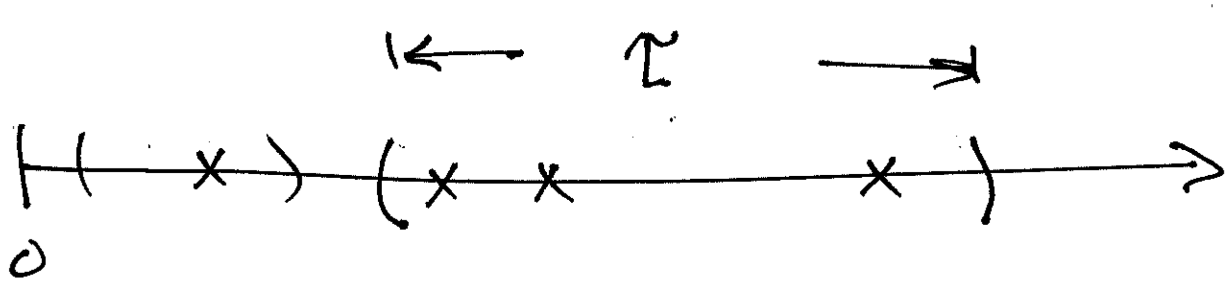
So any of the sequences:

$$X_t, N_t, Y_k, T_k$$

could be used to define a Bernoulli Process.

The Poisson Process is the continuous time limit of the Bernoulli Process.

Consider an arrival Process that evolves in continuous time



Define

$$P(k, \tau) = P(\#\text{ arrivals in an interval of length } \tau \text{ is } k)$$

Defn

An arrival Process is a Poisson Process with rate λ iff it the Properties

(a) time homogeniety

$P(k, \tau)$ is the same for all intervals of length τ.

(b) independence

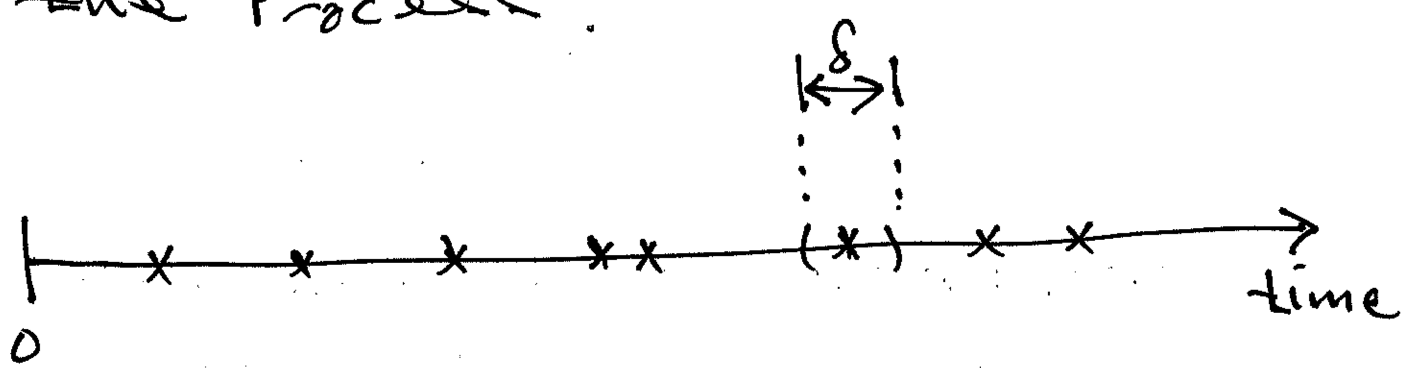
The number of arrivals in disjoint time intervals are independent events

(C) Small interval Probabilities

Let $\delta > 0$ be small. Then

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta & \text{if } k = 0 \\ \lambda\delta & \text{if } k = 1 \\ 0 & \text{if } k = 2, 3, \dots \end{cases}$$

Here λ is the arrival rate of the Process.



The approximation $\approx$ refers to limit as $\delta \to 0$.

To be precise

$$P(k, \delta) = \begin{cases} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k=2,3\ldots \end{cases} + o(\delta)$$

where $o(\delta)$ denotes a function with the property

$$\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$$

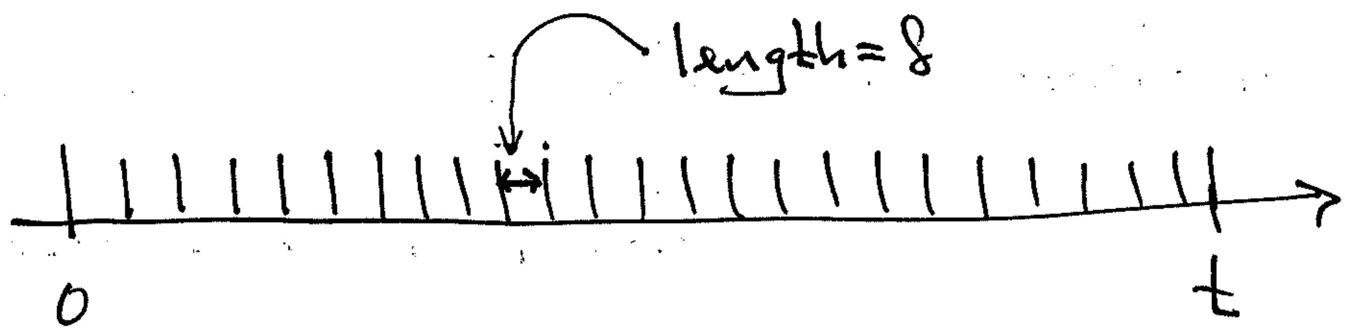
For instance $c \cdot \delta^2$ has this property. Observe

$E[\#\text{arrivals in } [t, t+\delta]]$

$$\approx (1-\lambda\delta)\cdot 0 + \lambda\delta \cdot 1 + 0 \cdot (\ldots) = \lambda\delta$$

Thus λ is # of arrivals per unit time.

What about a large time interval.



say from 0 to t : $[0,t]$. so

the # of small intervals is

$$n = \frac{t}{\delta}$$

i.e. $\delta n = t$, $\delta = \frac{t}{n}$.