CSE 107 3-5-25

Recall:

T_1 = time to 1st arrival

T_2 = " from T_1 to 2nd arrival

T_3 = " " T_2 " 3rd "

$\vdots$

Then

Y_K = time to K^{th} arrival

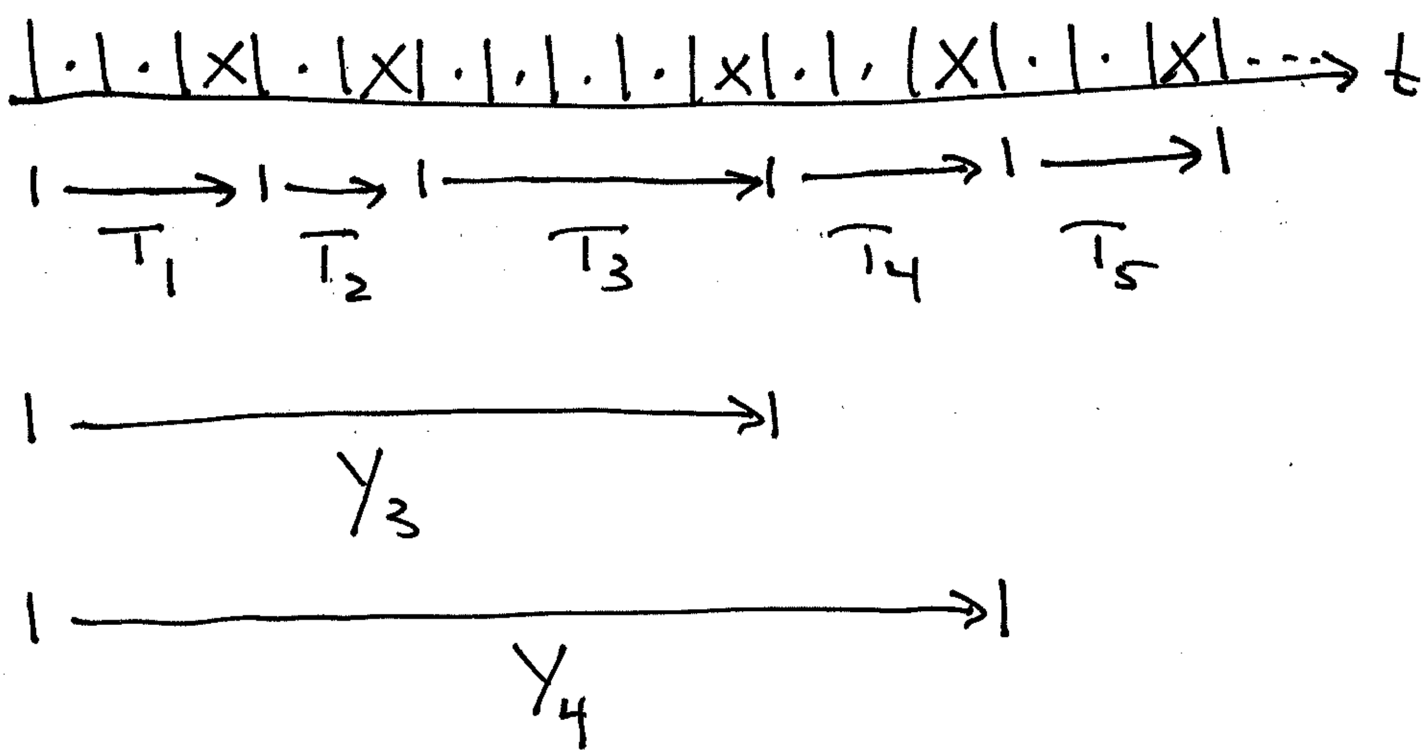
$= T_1 + T_2 + \cdots + T_{K-1} + T_K$

Also

$$\begin{cases} Y_1 = T_1 \\ Y_K = Y_{K-1} + T_K \qquad (K \geq 2) \end{cases}$$

Also we have

$$\begin{cases} T_1 = Y_1 \\ T_k = Y_k - Y_{k-1} & (k \geq 2) \end{cases}$$



By memorylessness the T_i are independent Geometric r.v., param. p.

What is the distribution of the Y_k $(k=1,2,3,\ldots)$?

<u>Exercise</u>

use convolution to find PMF of

$$Y_2 = Y_1 + T_2$$

$$Y_3 = Y_2 + T_3$$

$$Y_4 = Y_3 + T_4$$

$$\vdots$$

$$Y_k = Y_{k-1} + T_k$$

$$\vdots$$

We call Y_k the Pascal random variable of order k.

Another way to find PMF $P_{Y_k}(t)$ of Y_k: use independence

<u>note</u>: $\text{range}(Y_k) = \{k, k+1, k+2, \ldots\}$

$$P_{Y_k}(t) = P(Y_k = t)$$

$$= P\left((k-1) \text{ arrivals in } [1, t-1], \begin{array}{c} 1 \text{ arrival} \\ \text{at time } t \end{array}\right)$$

$$= \underbrace{P\left((k-1) \text{ arrivals in } [1, t-1]\right)}_{\text{Binomial}(t-1,\, p;\, k-1)} \cdot \underbrace{P\left(\begin{array}{c} 1 \text{ arrival} \\ \text{at time } t \end{array}\right)}_{p}$$

$$= \binom{t-1}{k-1} p^{k-1} \cdot (1-p)^{(t-1)-(k-1)} \cdot p$$

$$= \binom{t-1}{k-1} p^{k} (1-p)^{t-k}$$

Thus

$$P_{Y_k}(t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k} & (t=k, k+1, \ldots) \\ 0 & \text{otherwise} \end{cases}$$

note

$$E[Y_k] = E[T_1 + \cdots + T_k] = \boxed{\frac{k}{p}}$$

and

$$Var(Y_k) = Var(T_1 + \cdots + T_k) \quad \{\text{by indep.}$$

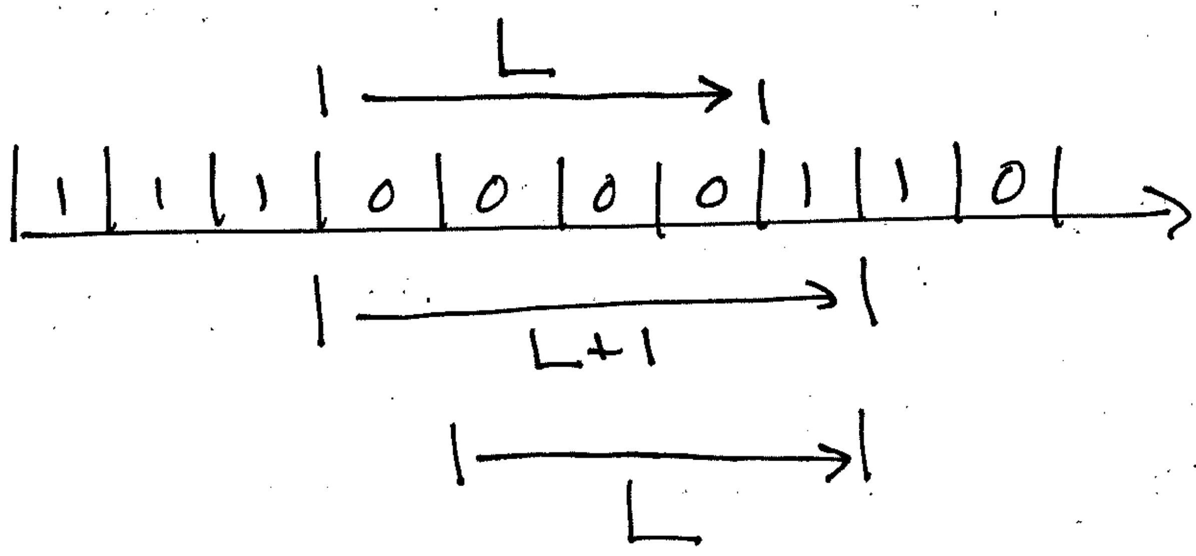
$$= Var(T_1) + \cdots + Var(T_k) = \boxed{\frac{k(1-p)}{p^2}}$$

Exercise check that

$$\sum_{t=k}^{\infty} P_{Y_k}(t) = 1$$

Ex

You Buy a lottery ticket every day. Let L be the length of your 1st losing streak.



find the distribution (i.e. PMF)

of L.

Soln

one might think $L+1$ is geometric, but its not!

why? L cannot be 0, so $L+1$ cannot be 1, so $L+1$ is not geometric. However

L = time from 2^{nd} loss until 1^{st} win.

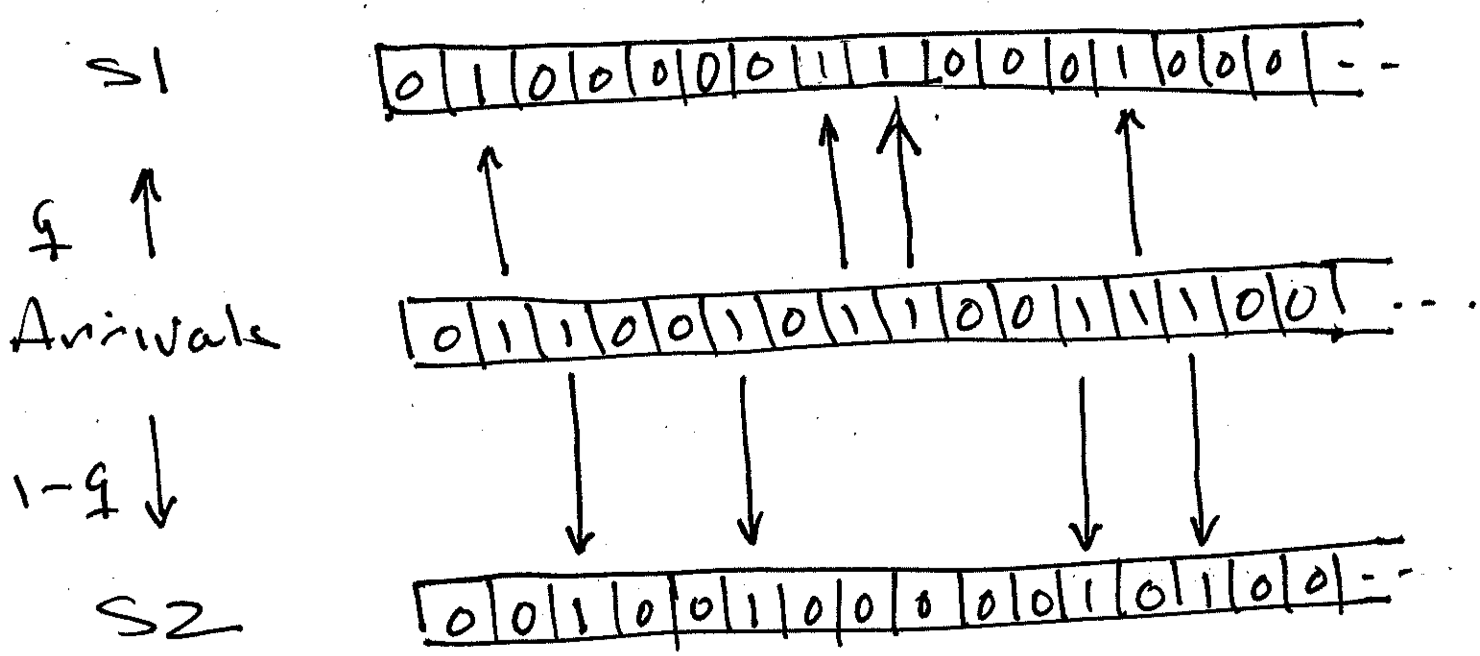
$\therefore$ L is geometric(p): $p(1-p)^{t-1}$

Splitting a Bernoulli Process

Suppose we have a Bernoulli Process with Param. p which produces a stream of arrivals of jobs into a queue. We send each job to one of 2 servers S_1, S_2

$$P(\text{Job goes to } S1) = q$$

$$P(\text{" " " } S2) = 1-q$$



Assume coin deciding the split is independent of the arrivals. So

P(arrival at S1) = $p \cdot q$

and

P(arrival at S2) = $p(1-q)$

Are S1 and S2 Bernoulli Processes?

answer: yes!