CSE 107 3-3-25

Ex we have 2 coins

coin 1 : $P(\text{heads}) = p$
coin 2 : $P(\text{heads}) = q$

flip coin until 1st head, let N be number of flips. flip coin 2 N times, count # heads.

each flip is a Bernoulli trial with

$P(X_i) = p$

The # of heads in 2^{nd} seq. is

$$Y = X_1 + X_2 + \cdots + X_N$$

<u>note</u>: Y is 'like' a Binomial r.v. with Param. q and N.

X is a r.v. with common mean and Var as each X_i. So

$$E[N] = \frac{1}{p}, \quad Var(N) = \frac{1-p}{p^2}$$

also

$$E[X] = q, \quad Var(X) = q(1-q)$$

Therefore

$$E[Y] = E[N]\cdot E[X] = \frac{q}{p}$$

and

$$Var(Y) = E[N]\,Var(X) + E[X]^2\,Var(N)$$

$$= \frac{q(1-q)}{p} + \frac{q^2(1-p)}{p^2}$$

$$= \frac{q}{p}\left((1-q) + \frac{q}{p}(1-p)\right)$$

6.1 The Bernoulli Process

<u>Defn</u>

The Bernoulli Process is a sequence

$$X_1, X_2, \ldots\ldots$$

of independent Bernoulli r.v.s, with

$$P(X_t = 1) = p \quad \begin{cases} \text{success, heads,} \\ \text{arrival} \end{cases}$$

and

$$P(X_t = 0) = 1-p \quad \begin{cases} \text{failure, tails,} \\ \ldots \end{cases}$$

Two basic questions

① Given the number of trials n performed, how many successes occurred?

ans: Binomial(n, p)

② how many trials until 1^{st} success

ans: Geometric(p).

Note: 2 ways to think of the infinite sequence

- one experiment, repeated ∞-many times
  $$\Omega = \{0,1\}$$
- a single experiment with
  $$\Omega = \{\text{infinite bit strings}\}$$

In the 2^{nd} view, with

$$0 < p < 1$$

let

$\omega = 1111 \cdots\cdots 111 \cdots\cdots$ (all 1's)

we have

$$P(\omega) = P(\forall t : X_t = 1)$$

$$\leq P(X_t = 1 \text{ for } t = 1, \ldots, n)$$

$$= p^n \quad (\text{for any } n \geq 1)$$

$$\therefore 0 \leq P(\omega) \leq p^n \longrightarrow 0 \text{ as } n \to \infty$$

$$\therefore P(\omega) = 0.$$

Memorylessness

Let T be the time (number of trials) to 1st success. So T is Geometric(p)

$$P_T(t) = (1-p)^{t-1} \cdot p \qquad (t = 1, 2, \ldots)$$

$$E[T] = \frac{1}{p}$$

$$Var(T) = \frac{1-p}{p^2}$$

Suppose $n \geq 1$ and

$$X_1, \ldots, X_n$$

are all failures. Then

$$\{T > n\}$$

has occurred. What is the dist. of $T-n$, the time left until the 1st head?

$$P(T-n=t \mid T>n) = \frac{P(T-n=t,\ T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t)}{P(T>n)}$$

since $\{T=n+t\} \subseteq \{T>n\}$

$$= \frac{(1-p)^{n+t-1} \cdot p}{(1-p)^n}$$

$$= \frac{\cancel{(1-p)^n} \cdot (1-p)^{t-1} \cdot p}{\cancel{(1-p)^n}}$$

$$= (1-p)^{t-1} \cdot p = P(T=t)$$

so independence implies memorylessness, i.e. "fresh start".

Inter-arrival times

Let

T_1 = time to 1^{st} arrival.

T_2 = time from T_1 to 2^{nd} arrival

T_3 = " " T_2 to 3^{rd} "

$\vdots$

Let

$$Y_K = T_1 + T_2 + \dots + T_K$$

$$= \text{time to } K^{th} \text{ arrival.}$$