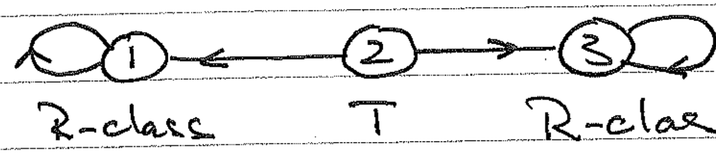
CSE 107 3-14-25

- Final: Th 3/20 4-6 PM
- supplemental lec. yesterday
- SETS: Due Sunday 3/16 11:59 PM

7.3 Steady-state behavior

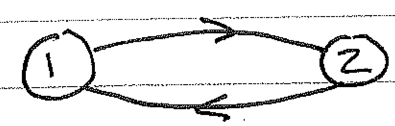
We do not expect the n-step transition probabilities to have limits (not depending on initial state) when there are either

- multiple recurrent classes



or

- periodic recurrent classes



We will write

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j \quad \text{(for all } i\text{)}$$

when the limit exists, and we call π_j the <u>steady-state probability</u> of j.

In the case where these limits exist (independent of i) we interpret

$$\pi_j \approx P(X_n = j) \quad \text{(for } n \text{ large)}$$

When do the π_j exist?

<u>Theorem</u> (steady-state convergence)
Consider a Markov chain with a single recurrent class, which is aperiodic. Then for each $j \in S$:

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j \quad \text{(for all } i\text{)}$$

exists. Furthermore, π_j are solutions to the linear system

(1) $$\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j \qquad (j = 1, 2, \ldots, m)$$

(2) $$\sum_{k=1}^{m} \pi_k = 1$$

Also $\pi_j = 0$ if j is transient, and $\pi_j > 0$ if j is recurrent.

Equations (1) are called the <u>balance equations</u>, and follow directly from the Chapman-Kolmogorov equations

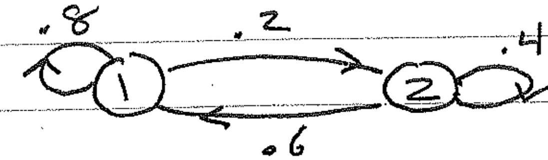
$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$

By taking limits $r_{ij}(n) \to \pi_j$ and $r_{ik}(n-1) \to \pi_k$ as $n \to \infty$.

Equations (2) are called the <u>normalization equations</u>, and follow from the fact that

$$\sum_{j=1}^{m} P(X_n = j) = 1 \quad (\text{any } n \geq 0)$$

Ex



$p_{11} = .8$, $p_{12} = .2$

$p_{21} = .6$, $p_{22} = .4$

$$(1) \quad \begin{cases} \pi_1 p_{11} + \pi_2 p_{21} = \pi_1 \\ \pi_1 p_{12} + \pi_2 p_{22} = \pi_2 \end{cases}$$

$$(2) \quad \pi_1 + \pi_2 = 1$$

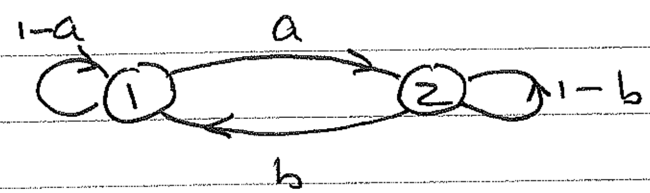
$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 = \pi_1 \iff (.6)\pi_2 = (.2)\pi_1 \\ (.2)\pi_1 + (.4)\pi_2 = \pi_2 \iff (.2)\pi_1 = (.6)\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

The 1st two give $\pi_1 = 3\pi_2$. The 3rd yields

$$3\pi_2 + \pi_2 = 1 \to 4\pi_2 = 1 \to \boxed{\pi_2 = \frac{1}{4}}$$

hence $\boxed{\pi_1 = \frac{3}{4}}$

Ex.



$$\pi_1(1-a) + \pi_2 b = \pi_1$$
$$\pi_1 a + \pi_2(1-b) = \pi_2$$

$$\pi_2 b = \pi_1 a$$

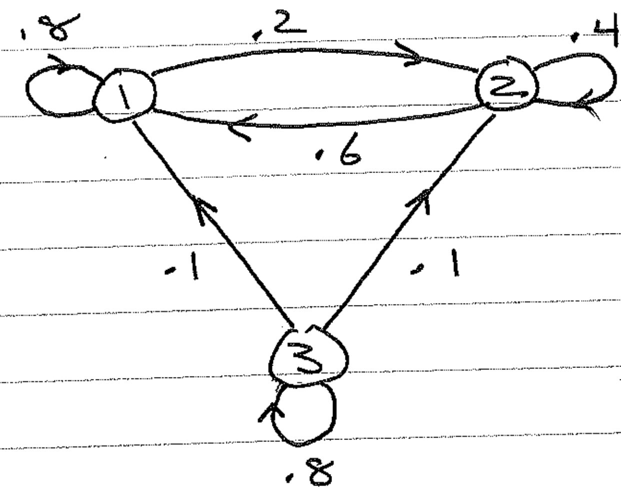
$$\therefore \quad \pi_1 = \frac{b}{a}\pi_2$$

$$\therefore \quad \frac{b}{a}\pi_2 + \pi_2 = 1$$

$$\therefore \quad \pi_2 = \frac{1}{\frac{b}{a}+1} \to \boxed{\pi_2 = \frac{a}{a+b}}$$

$$\therefore \quad \boxed{\pi_1 = \frac{b}{a+b}}$$

Ex.



$$\begin{cases} \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} = \pi_2 \\ \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} = \pi_3 \end{cases}$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 + (.1)\pi_3 = \pi_1 \\ (.2)\pi_1 + (.4)\pi_2 + (.1)\pi_3 = \pi_2 \end{cases}$$

$$\begin{cases} -2\pi_1 + 6\pi_2 + \pi_3 = 0 \\ 2\pi_1 - 6\pi_2 + \pi_3 = 0 \end{cases} \rightarrow 2\pi_3 = 0 \rightarrow \boxed{\pi_3 = 0}$$

$$\pi_1 = 3\pi_2$$

$$3\pi_2 + \pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}} \quad \therefore \boxed{\pi_1 = \frac{3}{4}}$$

See examples 7.6 and 7.7 on Pages 355 and 356, resp.

we should have anticipated this since we know state 3 is transient