CSE 107 3-13-25

Supplemental Lecture

- final exam: Th 3/20 4-6PM
- SETs: see offer on Ed.

Ex.
Returning to the previous example, we have

$$R^3 = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}\begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix} = \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}$$

so $P(X_3 = 1 \mid X_0 = 2) = r_{21}(3) = .744$. One calculates with some effort

$$R^{10} = \begin{pmatrix} .7500000256 & .2499999744 \\ .7499999232 & .2500000768 \end{pmatrix}$$

so that $P(X_{10} = 1 \mid X_0 = 2) = r_{21}(10) = .7499999232$.

It can be shown that

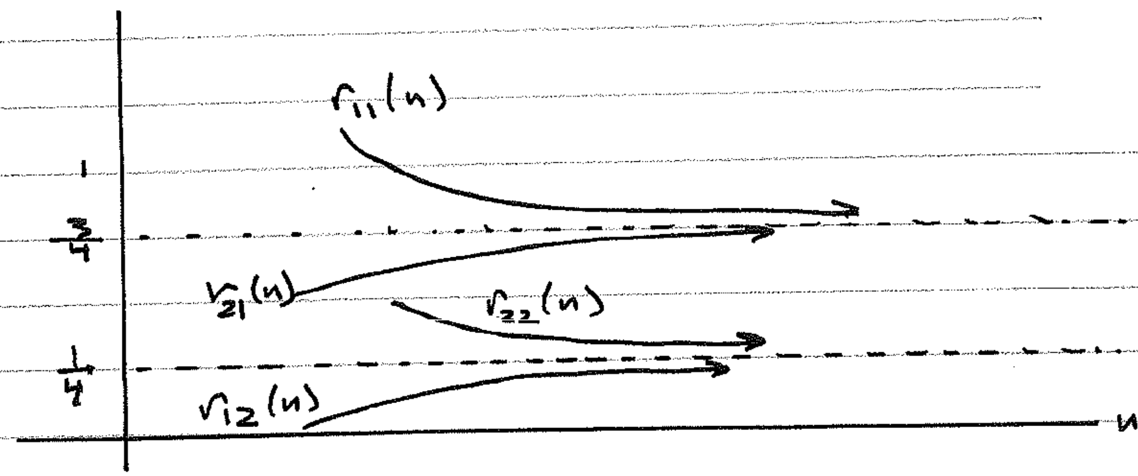
$$\lim_{n \to \infty} R^n = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix}$$

i.e.

$$r_{11}(n) \to \frac{3}{4}^{+} \qquad r_{12}(n) \to \frac{1}{4}^{-}$$

$$r_{21}(n) \to \frac{3}{4}^{-} \qquad r_{22}(n) \to \frac{1}{4}^{+}$$

as $n \to \infty$.

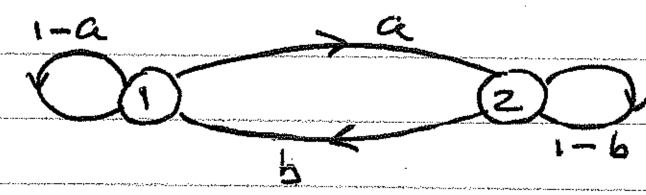


Thus, if we wait a long time, then

$$P(X_n=1) \approx \frac{3}{4} \text{ and } P(X_n=2) \approx \frac{1}{4}$$

independently of the initial state X_0. These are called <u>steady state probabilities</u>. Such behavior is not unusual in a Markov Chain.

<u>Ex.</u>



$$R = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

one can show (using linear algebra)

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

and hence

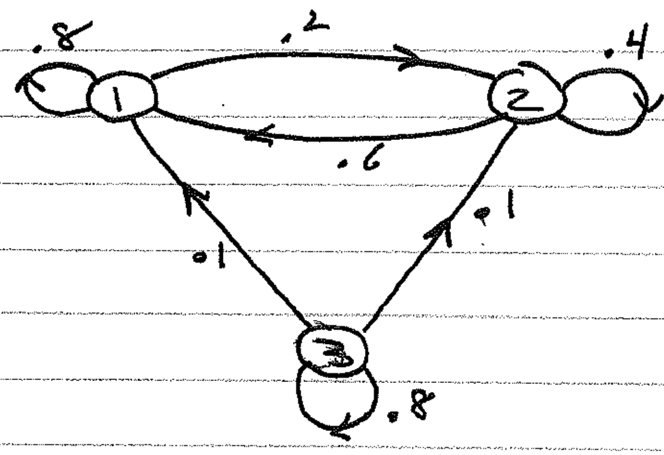
$$\lim_{n\to\infty} P(X_n = 1) = \frac{b}{a+b}$$

and

$$\lim_{n\to\infty} P(X_n = 2) = \frac{a}{a+b}$$

again regardless of X_0.

__Ex__



Thus

$$P = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .4 & 0 \\ .1 & .1 & .8 \end{pmatrix}$$

one can show

$$\lim_{n\to\infty} R^n = \begin{pmatrix} .75 & .25 & 0 \\ .75 & .25 & 0 \\ .75 & .25 & 0 \end{pmatrix}$$

hence

$$P(X_n = 1) \to \frac{3}{4} \quad \text{as } n \to \infty$$

$$P(X_n = 2) \to \frac{1}{4} \quad \text{as } n \to \infty$$

$$P(X_n = 3) \to 0 \quad \text{as } n \to \infty$$

regardless of the initial state X_0.

All of these limits can be computed using theorems of linear algebra, but is there another way that does not use L.A. topics? yes.

7.2 Classification of states

Defn
We say that state j is *accessible* from state i (or that j is *reachable* from i) iff

$$r_{ij}(n) > 0 \quad \text{for some } n \geq 1$$

Equivalently, there exists a sequence of states $i, i_1, i_2, \ldots, i_{n-1}, j$ such that the transitions

$$i \to i_1 \to i_2 \to \cdots \to i_{n-1} \to j$$

all have positive probability.

notation: $i \dashrightarrow j$

Defn
Let $A(i)$ denote the set of states accessible from i, i.e.

$$A(i) = \{ j \in S \mid i \dashrightarrow j \}$$

Defn

A state $i \in S$ is called recurrent if for every j accessible from i, we also have i accessible from j.

Equivalently

$$j \in A(i) \text{ iff } i \in A(j)$$

i.e.

$$(i) \dashrightarrow (j) \text{ iff } (j) \dashrightarrow (i)$$

Thus if we depart a recurrent state, there is a positive probability that we return, and so after a long enough time, we will return.

Hence a recurrent state, if it is visited at all, will be visited infinitely often.

Defn

A state i is called transient iff it is not recurrent. Thus i is transient iff there exists a state j with

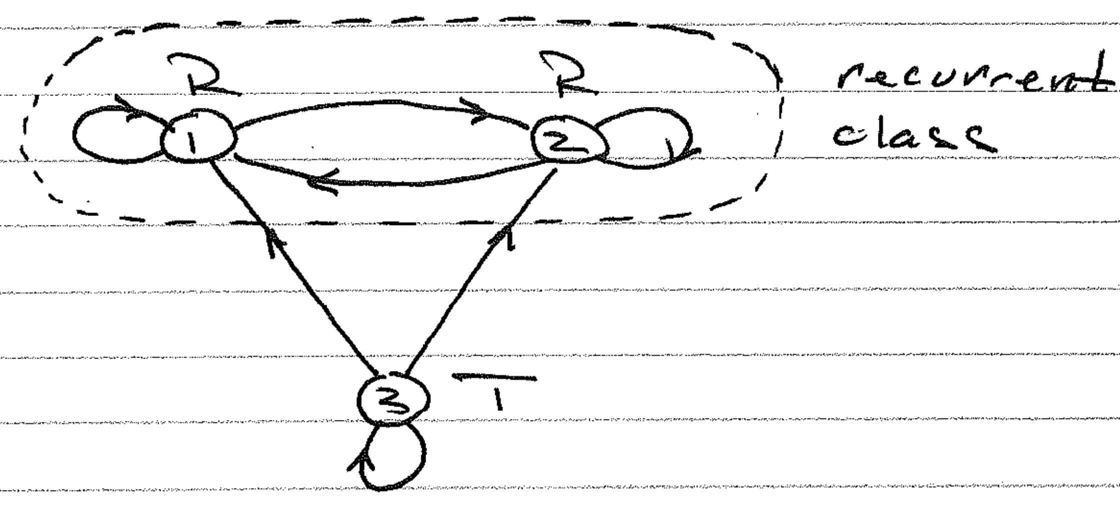
$$j \in A(i) \text{ and } i \notin A(j)$$

Thus after a visit to a transient state i, there is a positive probability of entering such a state j. Given enough time, this is certain to happen.

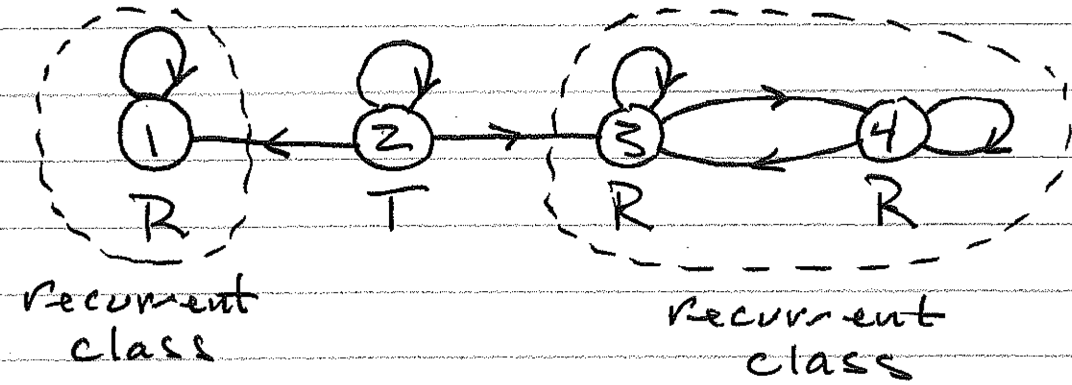
Hence a transient state will only be visited a finite number of times.

<u>Note</u>: transience and recurrence are determined by the arcs in a state transition diagram, not by the actual probabilities p_{ij} on each arc.

<u>Ex.</u>



<u>Ex.</u>



Note that for any states i, j we have

$$j \in A(i) \Rightarrow A(j) \subseteq A(i)$$

and

$$i \in A(j) \Rightarrow A(i) \subseteq A(j)$$

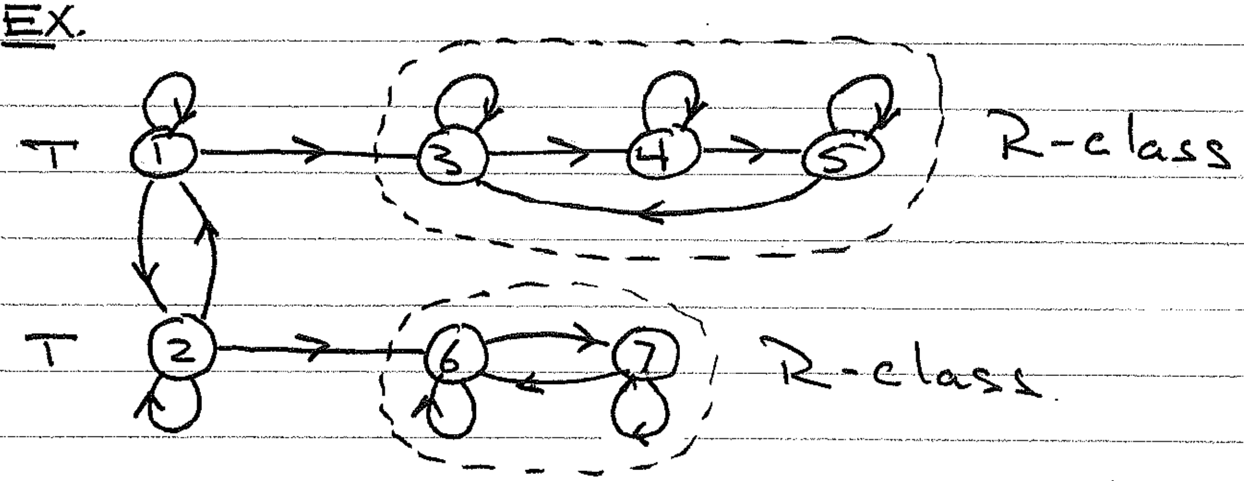
Therefore if i is recurrent, then

$$\left.\begin{array}{c} j \in A(i) \Rightarrow A(j) \subseteq A(i) \\ \Updownarrow \\ i \in A(j) \Rightarrow A(i) \subseteq A(j) \end{array}\right\} \Rightarrow A(i) = A(j)$$

<u>Defn</u>

If i is a recurrent state, the set $A(i) \subseteq S$ is called a <u>recurrent class</u>.

<u>EX.</u>



Problem
Exercise (see #6 on p.381)
Let i be a transient state. Prove there exists a recurrent state j such that

$$T \;\; i \dashrightarrow j \;\; R$$

i.e.

$$j \in A(i)$$

Theorem: Markov Chain Decomposition

- A Markov chain can be decomposed into one or more recurrent classes, and possibly some transient states.
- A recurrent state is accessible from all states in its class, but not from recurrent states in other classes.
- A transient state is not accessible from any recurrent state
- At least one recurrent state is accessible from a given transient state.

Theorem

(a) When a Markov chain enters a recurrent class, it stays within that class, and all states in the class will be visited infinitely often

(b) If the initial state of a Markov chain is transient, then its trajectory consists of a finite sequence of transient states, followed by an infinite sequence of recurrent states belonging to the same class.

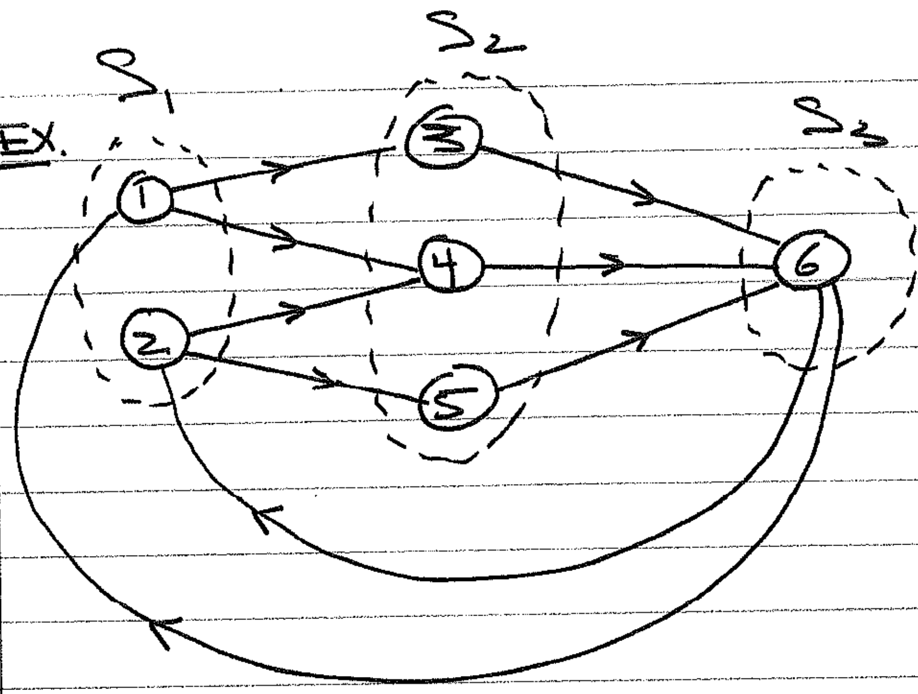
Defn

A recurrent class is called *Periodic* if its states can be grouped into $d \geq 2$ disjoint subsets $S_1, \ldots, S_d$ such that all transitions from S_k lead to S_{k+1} $(1 \leq k < d)$ and S_d leads to S_1.

$$\begin{tikzcd} S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \cdots \arrow[r] & S_{d-1} \arrow[r] & S_d \arrow[lllll, bend left] \end{tikzcd}$$

We call d the *period* of the class

EX.



Period = 3

observe that in this example, if $X_0 \in S_1$, then

$$X_n \in S_1 \quad \text{iff} \quad n \equiv 0 \pmod 3$$
$$X_n \in S_2 \quad \text{iff} \quad n \equiv 1 \pmod 3$$
$$X_n \in S_3 \quad \text{iff} \quad n \equiv 2 \pmod 3$$

Defn
A recurrent class is called aperiodic iff it is not periodic. Equivalently, there exists a time $n \geq 0$ such that

$$r_{ij}(n) > 0 \quad \text{for all} \quad i,j \in R$$

EX. add a self loop to any state in the previous example.