CSE 107 3-12-25

- Final exam : Th March 20, 4-6 PM
- SETs : closes Sun. March 16, 11:59 PM

If Response rate is ≥ 80%,

then will add 0.5% to all

overall scores.

7.1 Discrete Markov Chains

- state space

$$S = \{1, 2, 3, \ldots, m\}$$

- Let $X_0, X_1, X_2, \ldots, X_n, \ldots$ be a seq. of r.v. with values in S.

$$X_n = \text{state of process at time } n.$$

- Define

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

for any $i, j \in S$ and $n \geq 0$. Called <u>state transition probabilities</u>.

- Markov Property

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \ldots, X_0=i_0)$$

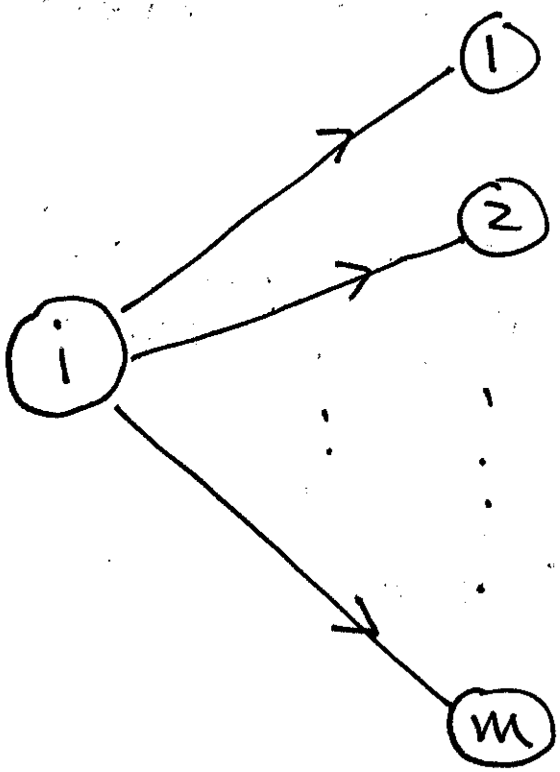
$$= P(X_{n+1}=j \mid X_n=i) = p_{ij}$$

These are ingredients of a

<u>Markov Chain Model</u>.

<u>Note</u>: we must have

$$\sum_{j=1}^{m} p_{ij} = 1 \quad (\text{for all } i \in S)$$

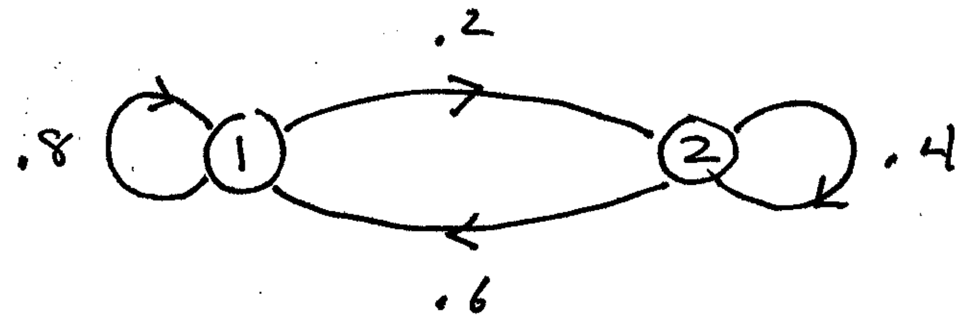


Ex. $m = 2$, $S = \{1, 2\}$

$p_{11} = .8$ $\quad$ $p_{12} = .2$

$p_{21} = .6$ $\quad$ $p_{22} = .4$

state transition diagram



Suppose $X_0 = 1$. Find $P(X_2 = 2)$

$$P(X_2=2) = P(X_2=2 \mid X_1=1)\cdot P(X_1=1)$$

$$+ P(X_2=2 \mid X_1=2)\cdot P(X_1=2)$$

$$= p_{12}\cdot P(X_1=1) + p_{22}\cdot P(X_1=2)$$

note

$$P(X_1=1) = P(X_1=1|X_0=1)\cdot\overbrace{P(X_0=1)}^{1}$$

$$+P(X_1=1|X_0=2)\cdot\overbrace{P(X_0=2)}^{0}$$

$$= p_{11}$$

and

$$P(X_1=2) = \cdots\cdots = p_{12}$$

Thus

$$P(X_2=2) = p_{12}\cdot p_{11} + p_{22}\cdot p_{12}$$

$$= p_{11}\cdot p_{12} + p_{12}\cdot p_{22}$$

$$= (.8)(.2) + (.2)(.4) = \boxed{.24}$$

Similarly, if $X_0 = 1$,

$$P(X_2=1) = P_{11} \cdot P_{11} + P_{12} \cdot P_{21}$$

$$= \boxed{.76}$$

In the same manner, if $X_0 = 2$, then

$$P(X_2=1) = P_{21} \cdot P_{11} + P_{22} \cdot P_{21} = \ldots = \boxed{.72}$$

$$P(X_2=2) = P_{21} P_{12} + P_{22} \cdot P_{22} = \ldots = \boxed{.28}$$

Summarize using transition probability matrix

$$R = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

Observe

$$R^2 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}^2 + p_{12}p_{21} & p_{11}p_{12} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{21} & p_{21}p_{12} + p_{22}^2 \end{pmatrix}$$

$$= \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

Defn

A Probability Matrix is a square $m \times m$ matrix in which each row sums to 1: i.e. if $A = (a_{ij})$ $1 \le i \le m$, $1 \le j \le m$ then

$$\sum_{j=1}^{m} a_{ij} = 1 \quad \text{for all } 1 \le i \le m.$$

Exercise

Prove that the Product of two Probability matrices, is a Probability matrix.

Defn

The <u>n-step transition Probabilities</u>

are

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

for any $i, j \in S$ and $n \geq 0$.

observe that

$$\boxed{r_{ij}(1) = p_{ij}}$$

Chapman-Kolmogorov equations

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj}}$$

for any $i, j \in S$, $n \geq 2$.

Matrix form

$$\boxed{\left(r_{ij}(n)\right)_{i,j \in S} = R^{n-1} \cdot R = R^{n}}$$

See proof in 7.1 of text.

$r_{ij}(n)$ is computed by conditioning on every possible predecessor of j