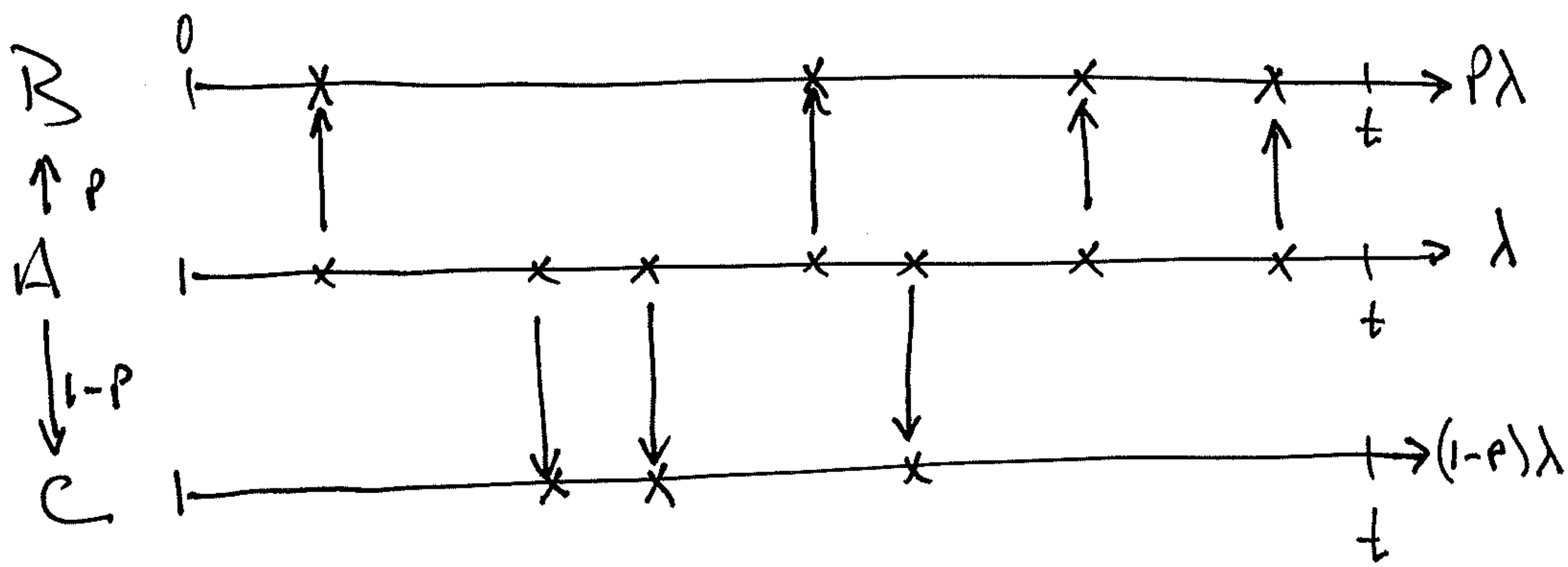
CSE 107 3-10-25

Splitting a Poisson Process

Let A be a Poisson Process of rate λ, and split A into two arrival processes B, C with Prob. p and $1-p$. (indep. of A)



Let

$N = \#$ arrivals in A in $[0,t]$

$X = \#$ " " B " "

$Y =$ " " " C " "

We know N is Poisson with param. λt

$$P_N(n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \quad (n=0,1,\dots)$$

Then

$$P_X(k) = P(X=k)$$

$$= \sum_{n=k}^{\infty} P(X=k \mid N=n)\cdot P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!\,(n-k)!} p^k (1-p)^{n-k} e^{-\lambda t} \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \cdot (\lambda t)^n}{(n-k)!}$$

sub. $n \leftarrow n+k$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)\lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\lambda t} \cdot e^{\lambda t - p\lambda t}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!}$$

$\therefore$ X is Poisson with Param. $p\lambda t$.

We see that B is a Poisson Process with rate $p\lambda$. Likewise C is a Poisson Process with rate $(1-p)\lambda$

Merging Poisson Processes

Ex.

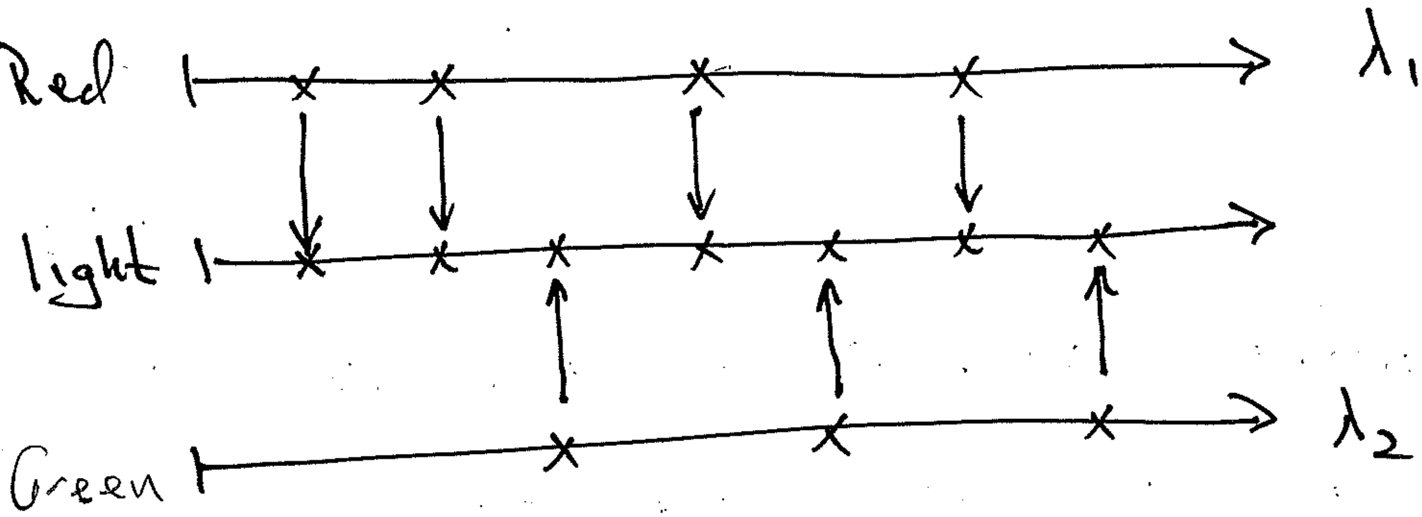
an observer sees two flashing light, one red, one green

Red light : Poisson Process rate λ_1

Green light : " " " λ_2

The observer is colorblind, and only sees a flashing light.

What kind of Process is the observer seeing ?



Let X = # red lights in $[0,t]$

Y = " Green " " "

M = # light " $[0,t]$

We know X is Poisson r.v. with Param. $\lambda_1 t$, and Y is a Poisson r.v. with Param. $\lambda_2 t$.

Thus $M = X + Y$ is a Poisson r.v. with parameter $\lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2) t$.

Thus the merged process is a Poisson process with rate $\lambda_1 + \lambda_2$.

<u>Ex.</u> (continuing)

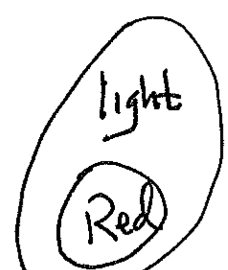
Suppose the observer sees a single flash. What's the probability that it was red?

Solution 1

We use the small interval probabilities. Let $\delta > 0$ be width of a small interval containing the flash. (Choose δ so small that Prob. of > 1 flash is negligible.) We seek

$$P(1 \text{ Red} \mid 1 \text{ light})$$

$$= \frac{P(1 \text{ Red},\ 1 \text{ light})}{P(1 \text{ light})}$$



$$= \frac{P(1 \text{ Red})}{P(1 \text{ light})}$$

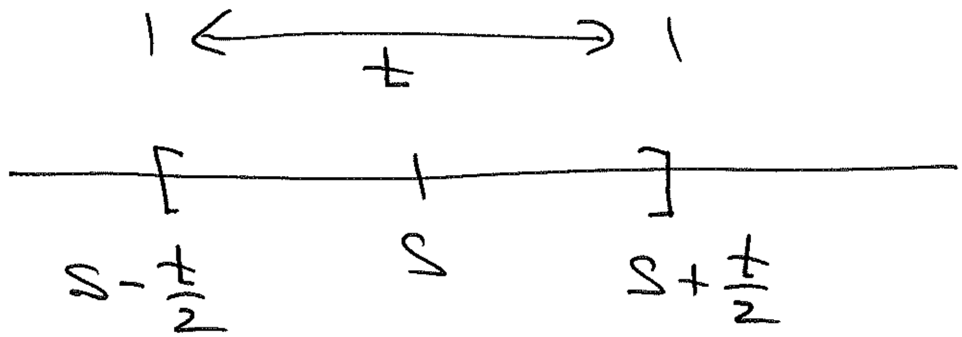
$$= \frac{\lambda_1 \cancel{\delta}}{(\lambda_1+\lambda_2)\cancel{\delta}} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Similarly, the Prob. that the light was green is $\dfrac{\lambda_2}{\lambda_1+\lambda_2}$.

<u>Solution</u> 2

not using small interval Prob.

suppose light flashes at time s. Consider:



let $I_t = \left[s - \frac{t}{2}, s + \frac{t}{2}\right]$.

Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t, 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{((\lambda_1+\lambda_2)t)^1}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[\text{as } t\to 0]{} \frac{\lambda_1}{\lambda_1+\lambda_2}$$