cse 107 2-7-25

Continuing CDF:

- note that

$$F_X(x) = P(X \leq x)$$

is always monotonically non-decreasing.

- if X is discrete $F_X(x)$ is piecewise constant.

$$F_X(k) = \sum_{i \leq k} P_X(i)$$

and

$$p_X(k) = P(X = k)$$

$$= P(X \le k) - P(X \le k-1)$$

$$= F_X(k) - F_X(k-1)$$

- If X is continuous, then $F_X(x)$ is a continuous function. Also

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

and

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Ex

You take a test 3 times, and final score X is the maximum of individual scores X_1, X_2, X_3

$$X = \max(X_1, X_2, X_3).$$

Assume X_i ($i=1,2,3$) are independent

Suppose X_i is uniform on $\{1, 2, \dots, 10\}$

i.e.

$$P_{X_i}(k) = \begin{cases} \frac{1}{10} & \text{if } k = 1, 2, \dots, 10 \\ 0 & \text{otherwise} \end{cases}$$

Find PMF of X.

First find CDF $F_X(k)$, then difference

$$P_X(k) = F_X(k) - F_X(k-1)$$

so

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

by indep. $\}$ $= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \frac{k}{10} \cdot \frac{k}{10} \cdot \frac{k}{10}$$

$$= \left(\frac{k}{10}\right)^3$$

Thus

$$P_X(k) = \begin{cases} \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 & \text{if } k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

<u>Goal</u>: Compare Geometric and Exponential CDFs

Recall if X is Geometric, param. p then

$$P_X(k) = p \cdot (1-p)^{k-1} \quad (k = 1, 2, \ldots)$$

Thus

$$F_{geo}(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n$$

If Y is Exponential, Param. λ, then

$$f_Y(x) = \lambda e^{-\lambda x} \qquad (x \geq 0)$$

Thus

$$F_{exp}(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x$$

$$= 1 - e^{-\lambda x}$$

To compare, Pick $\delta > 0$ such that

$$\boxed{e^{-\lambda\delta} = 1 - p}$$

i.e. $\quad p = 1 - e^{-\lambda\delta}$

i.e. $\quad -\lambda\delta = \ln(1-p)$

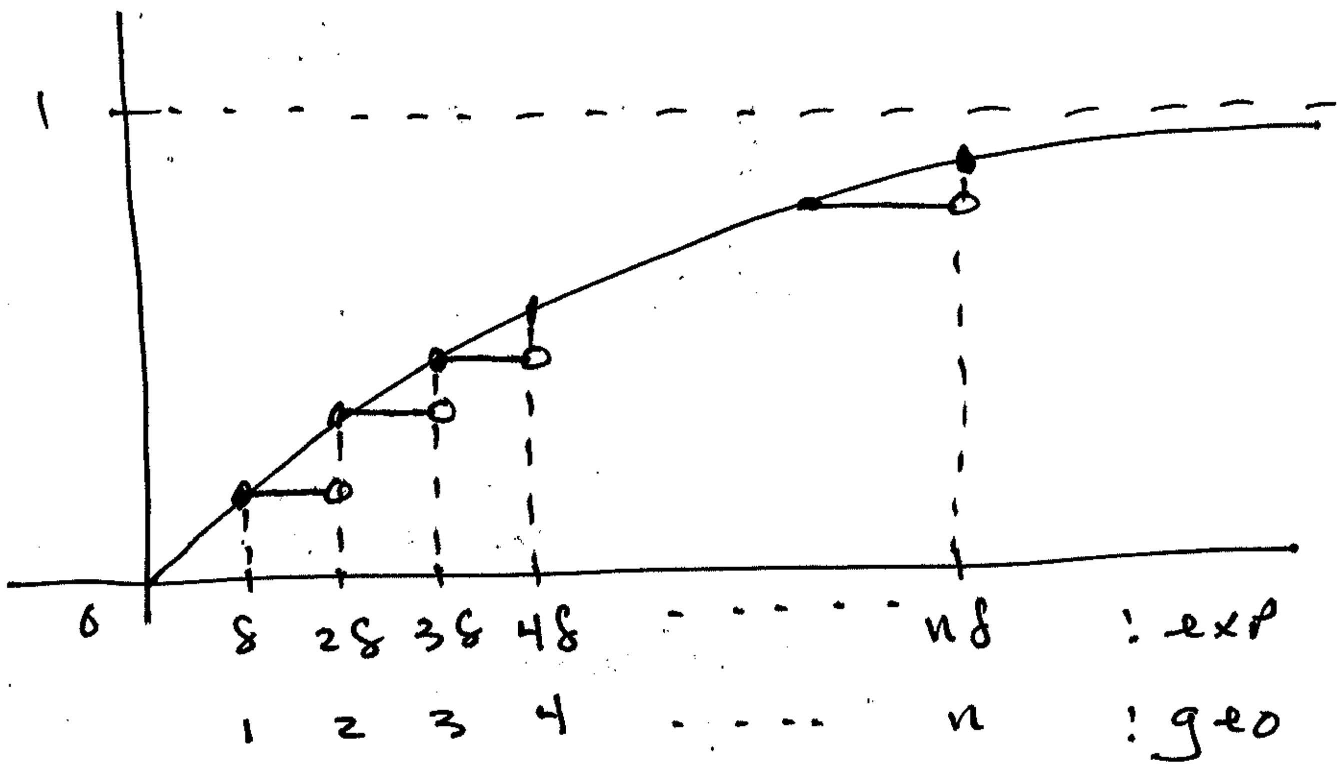
i.e. $\quad \delta = \dfrac{-\ln(1-p)}{\lambda}$

For this δ, we have

$$\left(e^{-\lambda\delta}\right)^n = (1-p)^n$$

$$\therefore \quad 1 - e^{-\lambda\delta n} = 1 - (1-p)^n$$

$$\therefore \quad F_{exp}(\delta n) = F_{geo}(n)$$



If we flip coin fast (every δ sec.) with Prob. $P = 1 - e^{-\lambda\delta}$, then the time of 1^{st} head is an approx to exponential CDF

3.3 Normal random Variables

Let X be a cont. r.v. with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

call this the normal or Gaussian r.v. with Params. σ, μ.

one checks

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Also

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$

$$\sqrt{Var(X)} = \sigma$$