CSE 107 2-6-25

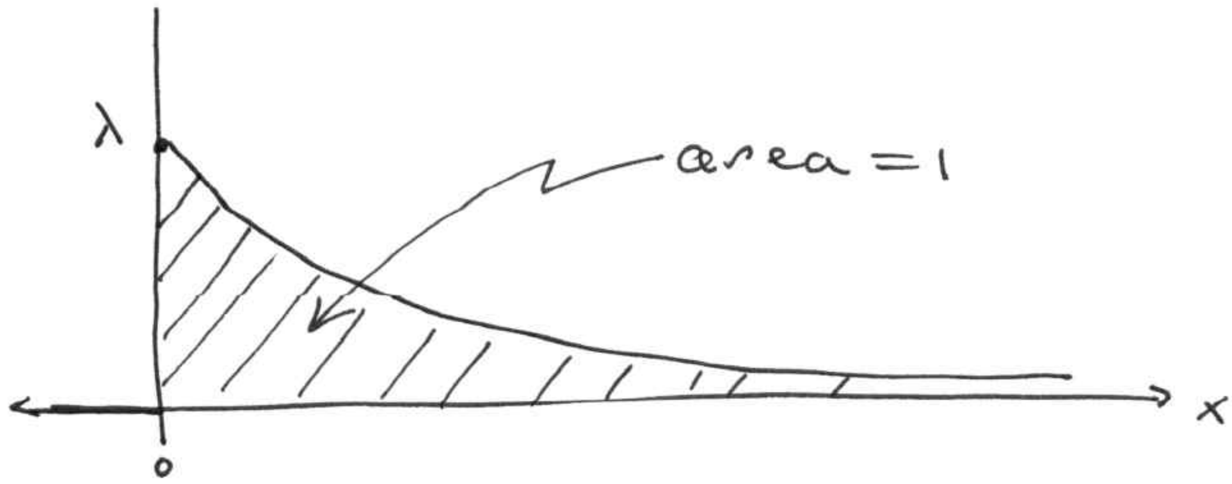
Supplemental Lecture

Ex. Exponential random variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

here $\lambda > 0$ is a parameter.

Exponential is continuous time analog of the Geometric. Recall: Poisson is the continuous time analog of the Binomial.



$$\int_{-\infty}^{\infty} f_X(x)\,dx = -\int_0^{\infty} -\lambda e^{-\lambda x}\,dx$$

$$= \; -e^{-\lambda x}\Big|_0^{\infty}$$

$$= -\lim_{a\to\infty}\left(e^{-\lambda a} - 1\right)$$

$$= -(0-1) = \boxed{1}$$

show that

- $E[X] = \frac{1}{\lambda}$ ✓

- $E[X^2] = \frac{2}{\lambda^2}$ (exercise)

- $Var(X) = \frac{1}{\lambda^2}$ ✓

<u>Proof</u>:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$= -\int_0^{\infty} x \left(-\lambda e^{-\lambda x}\right) dx$$

integration by Parts

$$u = x \quad dv = -\lambda e^{-\lambda x} dx$$
$$du = dx \quad v = e^{-\lambda x}$$

$$= -\left(x e^{-\lambda x} \Big|_0^{\infty} - \int_0^{\infty} e^{-\lambda x} dx \right)$$

$$= -\left(0 - \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty \right)$$

$$= -\frac{1}{\lambda}(0 - 1) = \frac{1}{\lambda} \checkmark$$

$$E[X^2] = -\int_0^\infty x^2 \left(-\lambda e^{-\lambda x}\right) dx$$

int. by Parts: $u = x^2 \quad dv = -\lambda e^{-\lambda x} dx$

$du = 2x\,dx \quad v = e^{-\lambda x}$

$$= \text{--- exercise ----} = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \checkmark$$

Lemma

If X is exponential, with Param. λ, then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \quad \checkmark \\ 1 & \text{if } a \leq 0 \quad \checkmark \end{cases}$$

Proof

let $a > 0$. Then

$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x}\, dx$$

$$= \lambda \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right)$$

$$= e^{-\lambda a} \qquad \blacksquare$$

Ex.

The time X until the next earthquake (4.0 or above) is modeled an exponential r.v. with mean 53 minutes.

What is the Probability that such a quake occurs in the next 10 minutes (starting now)?

Solution

$$E[X] = \frac{1}{\lambda} = 53 \Rightarrow \lambda = \frac{1}{53},$$

hence

$$P(0 \le X \le 10) = P(X \ge 0) - P(X > 10)$$

$$= 1 - e^{-\lambda \cdot 10}$$

$$= 1 - e^{-10/53}$$

$$= \boxed{.1719}$$

Exercise

determine a such that the Probability of an earthquake within the next a minutes is 0.99, i.e

$$P(0 \le X \le a) = .99$$

answer: $53 \cdot \ln(100) = 244.07$ minutes
$= 4.068$ hours.

Note: X is time to next event

$\therefore$ $E[X]$ has units $\frac{\text{time}}{\text{event}}$

$\therefore$ $\frac{1}{\lambda}$ " " $\frac{\text{time}}{\text{event}}$

$\therefore$ λ " " $\frac{\text{event}}{\text{time}}$

i.e. λ is the event rate or the arrival rate.

3.2 Cumulative Distribution Functions (CDF)

Defn

Let X a random variable (discrete or continuous.) The CDF of X is

$$F_X(x) = P(X \le x)$$

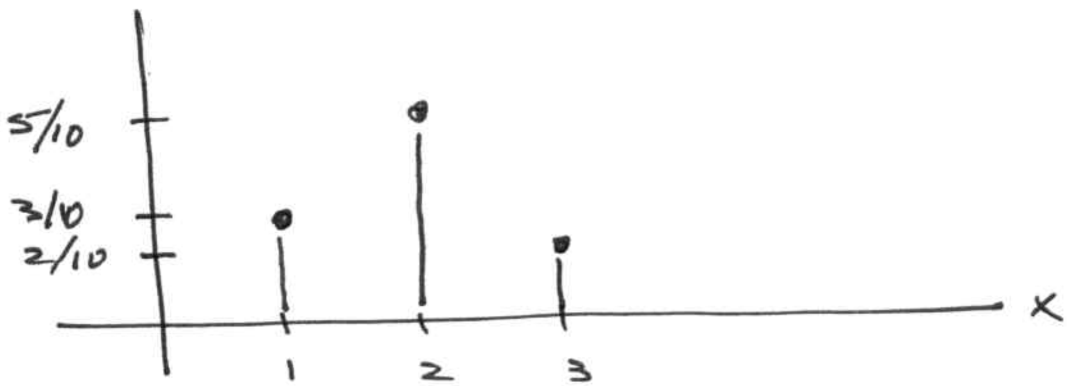
Note: The CDF is a unified representation of both discrete and continuous r.v.s.

$$F_X(x) = \begin{cases} \text{discrete: } \sum_{k \le x} P_X(k) \\ \text{continuous: } \int_{-\infty}^{x} f_X(t)\,dt \end{cases}$$

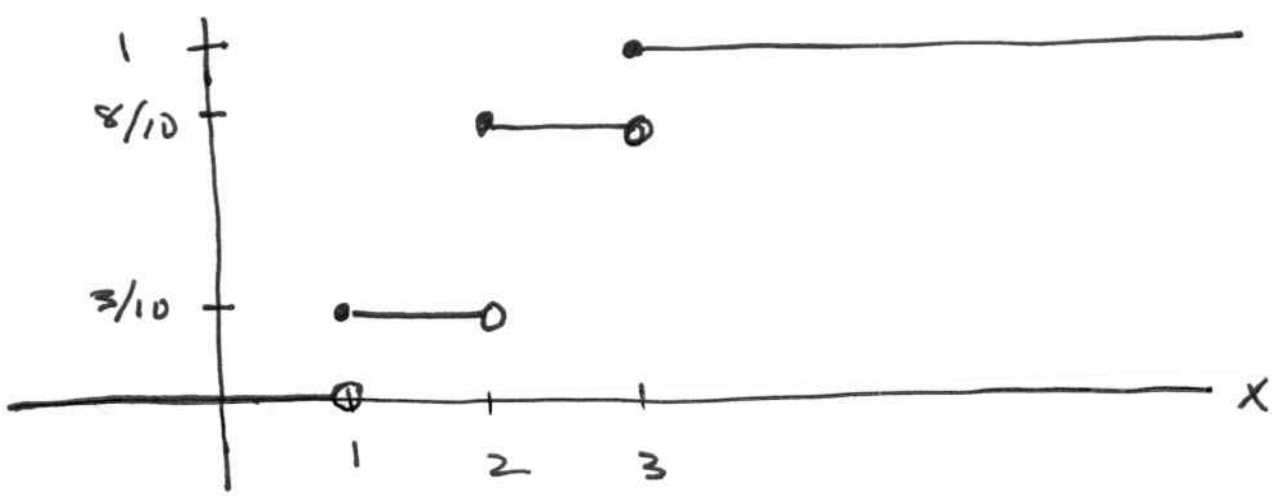
observe the CDF accumulates probability (mass) up to $x \in \mathbb{R}$.

Ex. (discrete)

PMF: $$P_X(k) = \begin{cases} 3/10 & \text{if } k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$$



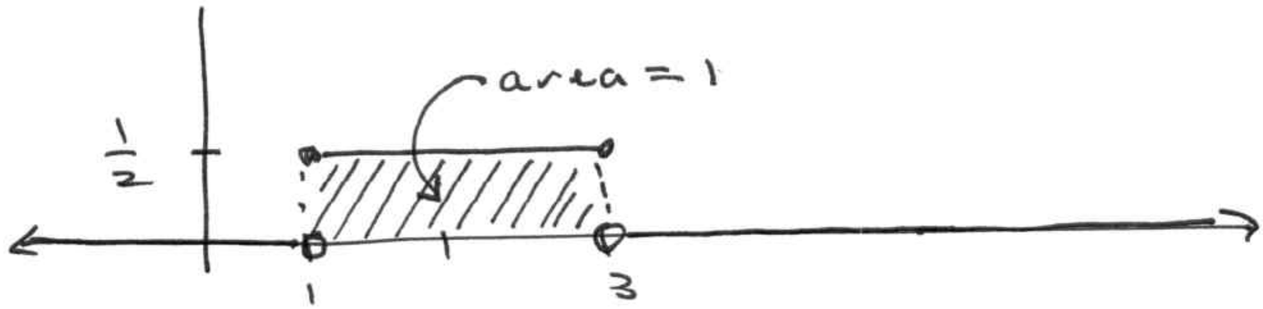
CDF: $$F_X(x) = \begin{cases} 0 & \text{if } x<1 \\ 3/10 & \text{if } 1 \le x < 2 \\ 8/10 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$



Ex (continuous)

uniform on $[1, 3]$

PDF: $f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$



CDF: $F_X(x) = \int_1^x \frac{1}{2}\,dt = \frac{1}{2}t\Big|_1^x = \frac{x-1}{2} \quad (1 \le x \le 3)$

$$F_X = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x-1}{2} & \text{if } 1 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$