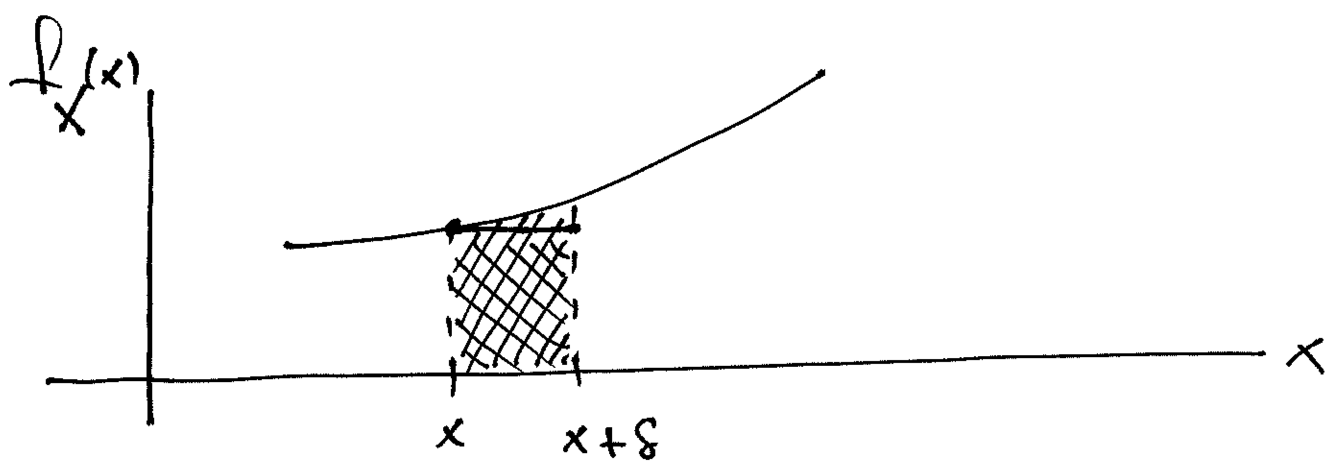
CSE 107 2-5-25

why is $f_X(x)$ called a density?

let $\delta > 0$ be small. suppose f_X is continuous in a neighborhood of x. Then f_X is approx. constant on $[x, x+\delta]$.



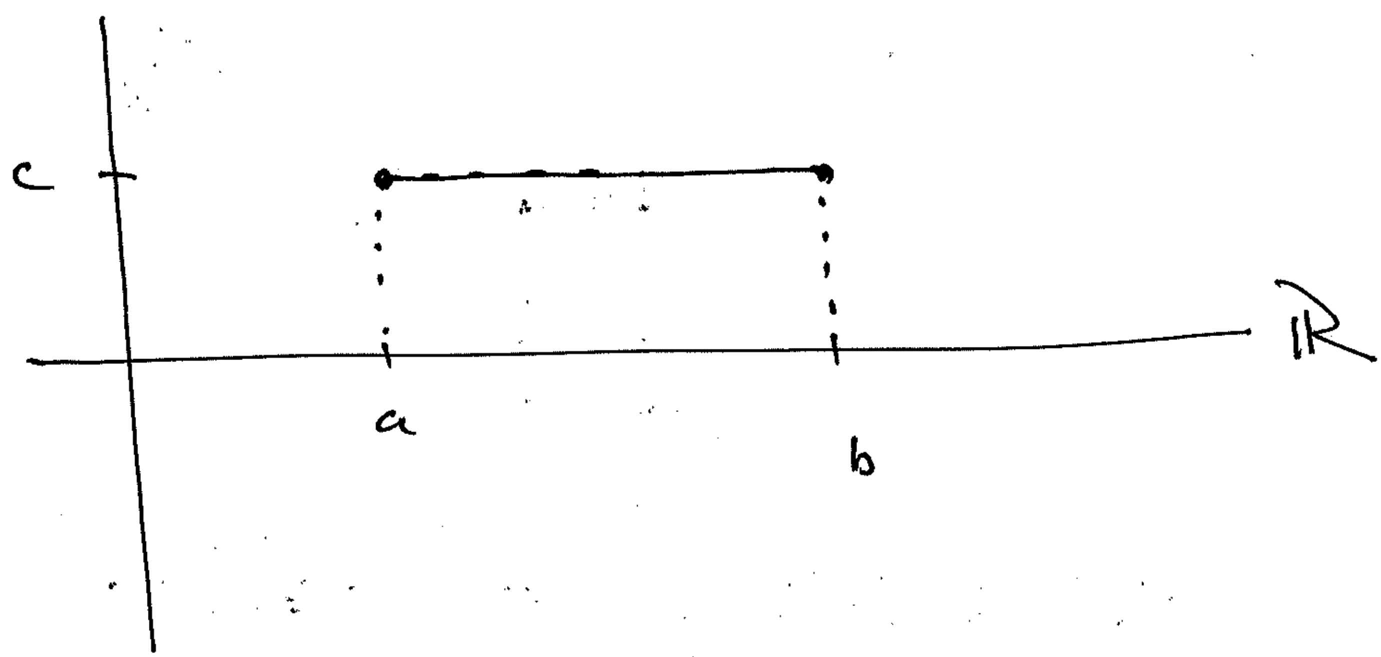
$$\underbrace{P(x \le X \le x+\delta)}_{\text{'mass'}} = \int_x^{x+\delta} f_X(t)\,dt \approx f_X(x) \cdot \underbrace{\delta}_{\text{length}}$$

so $f_X(x)$ has units $\frac{\text{mass}}{\text{length}}$

mass per unit length, i.e. density.

Ex continuous uniform r.v. on $[a,b]$.

PDF $$f_X(x) = \begin{cases} c & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



<u>note</u>:

$$1 = P(\Omega) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= \int_a^b c\,dx = c \cdot (b-a)$$

$$\therefore \quad c = \frac{1}{b-a}.$$

Thus

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Ex. Piecewise constant PDF.

Alice's driving time to work is 40-44 minutes in good weather, and 44-50 in bad, unif. distributed in each case. Suppose

$$P(\text{good weather}) = \frac{2}{3}$$

Find the PDF of X, Alice's driving time.

We have

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \le x \le 44 \text{ (good)} \\ c_2 & \text{if } 44 < x \le 50 \text{ (bad)} \\ 0 & \text{otherwise} \end{cases}$$

Also

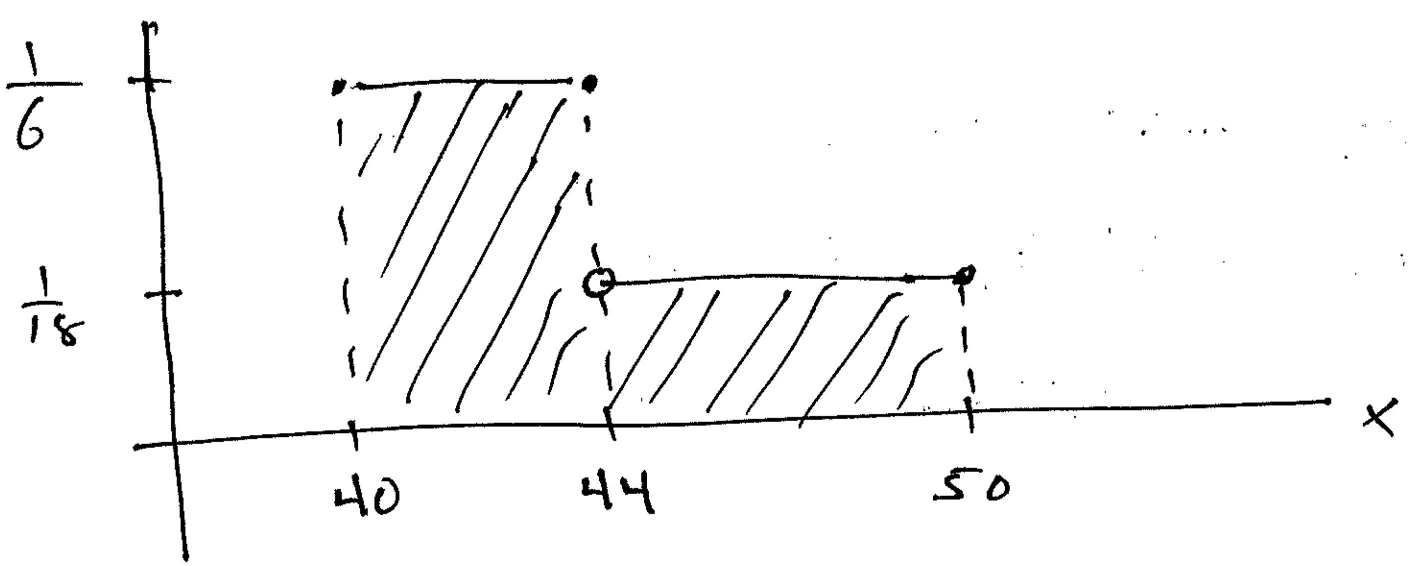
$$\frac{2}{3} = P(\text{good}) = \int_{40}^{44} c_1 \, dx = 4 \cdot c_1$$

$$\therefore \quad c_1 = \frac{1}{4} \cdot \frac{2}{3} = \frac{2}{12} = \boxed{\frac{1}{6}}$$

$$\frac{1}{3} = P(\text{bad}) = \int_{44}^{50} c_2 \, dx = 6 c_2$$

$$\therefore \quad c_2 = \frac{1}{3} \cdot \frac{1}{6} = \boxed{\frac{1}{18}}$$

Thus

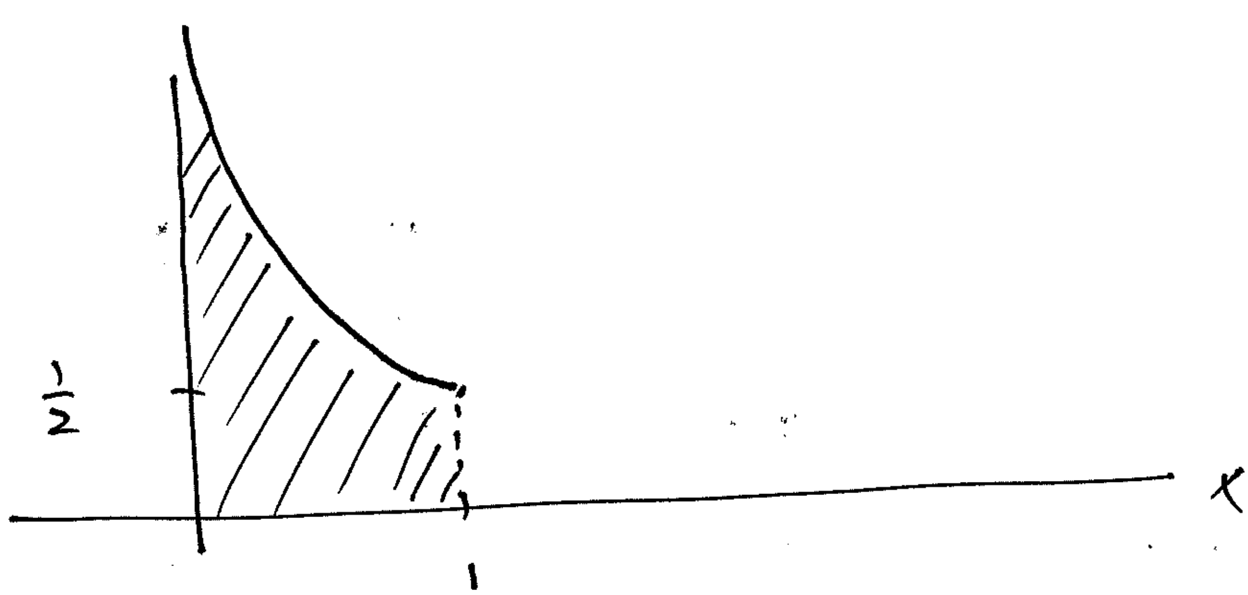


<u>note</u> a PDF $f_X(x)$ must be

- non-negative.
- total area $= 1$

However $f_X(x)$ need not be bounded above.

<u>Ex.</u> $$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 \tfrac{1}{2} x^{-\frac{1}{2}}\,dx$$

$$= \cancel{\tfrac{1}{2}} \lim_{a\to 0^+} \left.\frac{x^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}\right|_a^1$$

$$= \lim_{a\to 0^+} \left(\sqrt{1} - \sqrt{a}\right) = 1,$$

so this is a valid PDF.

$\boxtimes$

Defn

The expected value of a cont. r.v. X is given by

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

Expected value rule:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

see Problem #4 p. 185 with soln for a proof.

Defn

The n^{th} moment of X is

$E[X^n]$ for $n = 1, 2, \ldots$.

The variance of X is

$$Var(X) = E[(X - E[X])^2]$$

Theorem Variance in terms of moments.

$$Var(X) = E[X^2] - (E[X])^2.$$

<u>Proof</u>

Let $\mu = E[X]$. Then

$$Var(X) = E\left[(X-\mu)^2\right]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - 2\mu \int_{-\infty}^{\infty} x f_X(x)\, dx + \mu^2 \int f_X(x) dx$$

$$= E[X^2] - 2\mu \cdot \mu + \mu^2$$

$$= E[X^2] - (E[X])^2. \quad \blacksquare$$

If $Y = aX + b$, then

- $E[Y] = a\,E[X] + b$
- $Var(Y) = a^2\,Var(X)$

<u>Exercise</u> Prove this!

<u>Ex</u> cont. unif. on $[a,b]$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \int_a^b x \cdot \left(\frac{1}{b-a}\right) dx$$

$$= \left(\frac{1}{b-a}\right) \cdot \frac{1}{2} x^2 \Big|_a^b$$

$$= \left(\frac{1}{b-a}\right) \cdot \frac{1}{2} (b^2 - a^2)$$

$$= \frac{1}{2} \cdot \frac{1}{\cancel{(b-a)}} \cdot \cancel{(b-a)}(b+a)$$

$$= \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2 \cdot \left(\frac{1}{b-a}\right) dx$$

$$= \underset{\text{exercise}}{- - - - - - - -} = \frac{a^2+ab+b^2}{3}$$

Hence

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$\vdots$ exercise

$$= \frac{(b-a)^2}{12}$$

Ex Alice's driving time again

Recall

$$f_X(x) = \begin{cases} 1/6 & 40 \le x \le 44 \\ 1/18 & 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\left.\begin{aligned} E[X] &= \frac{131}{3} \\ E[X^2] &= \frac{17{,}228}{9} \\ Var(X) &= \frac{67}{9} \end{aligned}\right\} \text{Verify these!}$$