cse 107 2-26-25

mid 2: Friday 2/28

law of Iterated Expectation

$$E[X] = E[E[X|Y]]$$

similar process with Variance

define

$$h(y) = Var(X \mid Y=y)$$

$$= E[(X - E[X|Y=y])^2 \mid Y=y]$$

By substituting Y for y, we get a new r.v.

$$h(Y) = Var(X \mid Y)$$

$$= E\left[(X - E[X|Y])^2 \mid Y \right]$$

This is called <u>Conditional Variance</u>

<u>Law of total Variance</u>

$$Var(X) = E\left[Var(X \mid Y)\right] + Var\left(E[X|Y]\right)$$

<u>Proof</u>

By variance in terms of moments

$$Var(X|Y) = E[X^2|Y] - E[X|Y]^2$$

$$\therefore E[X^2|Y] = Var(X|Y) + E[X|Y]^2$$

By iterated expectation:

$$Var(X) = E[X^2] - E[X]^2$$

$$= E[E[X^2|Y]] - E[E[X|Y]]^2$$

$$= E\left[\mathrm{Var}(X|Y) + E[X|Y]^2 \right]$$

$$- E\left[E[X|Y]\right]^2$$

$$= E[\mathrm{Var}(X|Y)]$$

$$+ E\left[E[X|Y]^2\right] - E\left[E[X|Y]\right]^2$$

$$= E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y])$$

$\blacksquare$

4.5 the sum of a random number of random variables

Let $X_1, X_2, \ldots$ be an (infinite) sequence of independent identically distributed (i.i.d) r.v.s.

This means they all have the same PDF or PMF, which we denote

$$f_X \quad \text{or} \quad p_X$$

Let N be a random variable with $\text{range}(N) \subseteq \{1, 2, 3, \ldots\}$

Assume

$$\{N, X_1, X_2, X_3, \ldots\}$$

is independent. Now define

$$Y = X_1 + X_2 + \cdots + X_N$$

We denote by $E[X]$, $Var(X)$ the common mean & variance of all X_i

Let $n \geq 1$ be fixed. Then

$$E[Y \mid N = n]$$

$$= E[X_1 + \cdots + X_N \mid N = n]$$

$$= E[X_1 + \cdots + X_n \mid N = n]$$

$$= E[X_1 + \cdots + X_n]$$

$$= n \cdot E[X]$$

Therefore

$$E[Y \mid N] = N \cdot E[X]$$

By iterated expectation

$$E[Y] = E[E[Y|N]]$$

$$= E[N \cdot E[X]]$$

$$= E[N] \cdot E[X]$$

Also

$$Var(Y|N=n)$$

$$= Var(X_1 + \cdots + X_N | N=n)$$

$$= Var(X_1 + \cdots + X_n | N=n)$$

$$= Var(X_1 + \cdots + X_n) \quad \Big\{ \text{by indep.}$$

$$= Var(X_1) + \cdots + Var(X_n) \quad \Big\{ \text{by indep.}$$

$$= n \cdot Var(X)$$

replace n by N to get

$$Var(Y|N) = N \cdot Var(X)$$

By law of total variance

$$Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$$

$$= E[N \cdot Var(X)] + Var(N \cdot E[X])$$

$$= E[N] \cdot Var(X) + E[X]^2 \cdot Var(N)$$

so

$$\boxed{E[Y] = E[N] \cdot E[X]}$$

$$\boxed{Var(Y) = E[N] Var(X) + E[X]^2 Var(N)}$$

Ex. (lab 4)

we have 2 coins

coin 1: $P(\text{head}) = p$

coin 2: $P(\text{head}) = q$

flip coin 1 until 1^{st} head, let # flips be N. (geometric with param. p).

filp coin 2 N times. Let Y be the # of heads, i.e.

$$Y = X_1 + \cdots + X_N$$

where each X_i is Bernoulli with param q.