CSE 107 2-22-25

Supplemental Lecture

Let X, Y be independent continuous r.v.s and $Z = X + Y$. Then

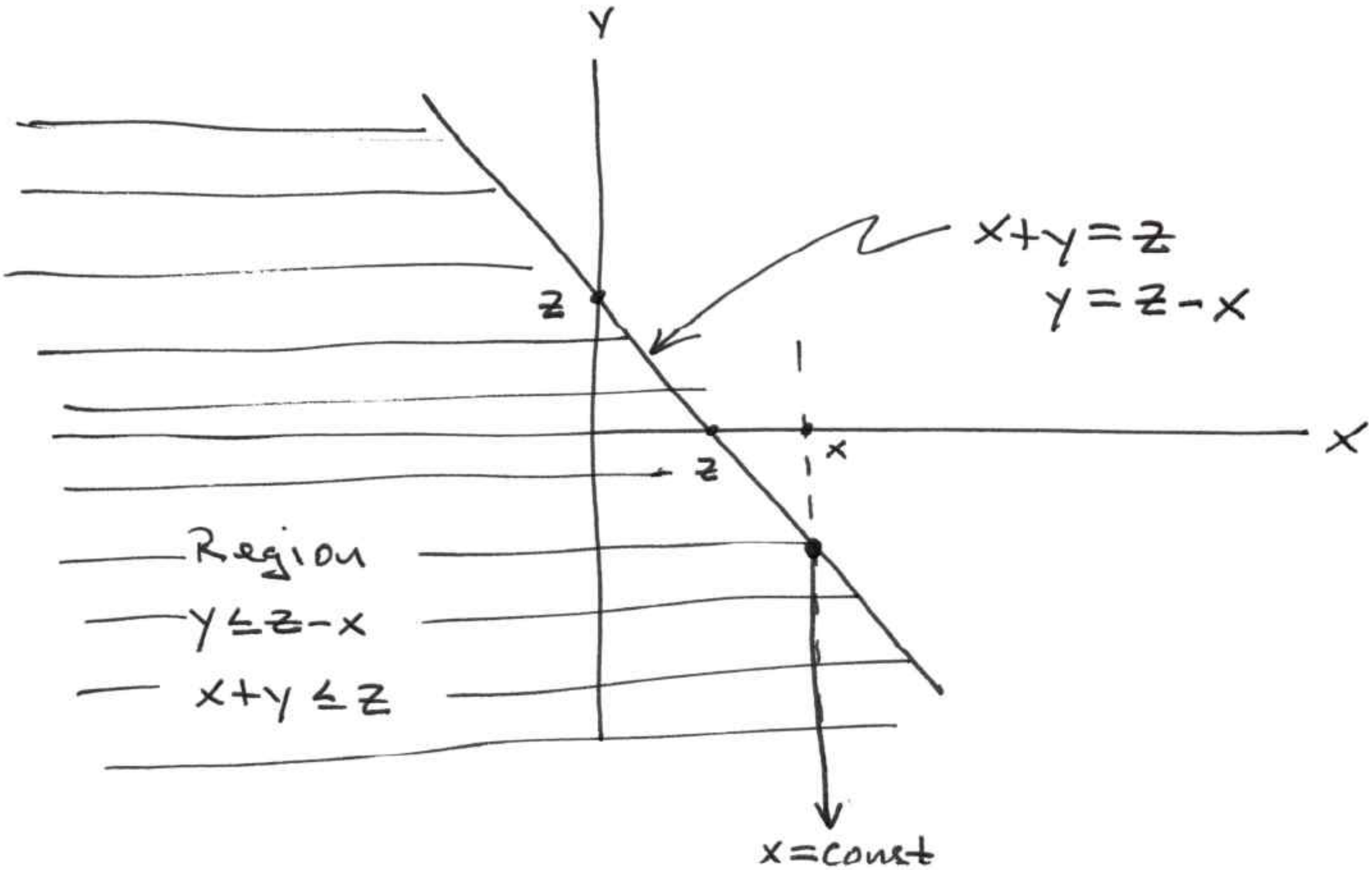
$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

$$= \iint_{\{(x,y) \mid x+y \le z\}} f_{X,Y}(x,y)\, dy\, dx$$

$(y \le z - x \text{ or } x \le z - y)$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) \cdot f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right) dx$$



$$= \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\, dx$$

differentiate w.r.t. z to get

$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) \cdot F_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \frac{d}{dz} F_Y(z-x) \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x) \, dx$$

This operation is called the convolution product of f_X and f_Y : $f_X * f_Y$

$$\boxed{f_{X+Y}(z) = \left(f_X * f_Y\right)(z)}$$

<u>note</u>: $f_X * f_Y = f_Y * f_X$.

Remark:

Remember, both discrete & contuous formulas for p_{X+Y} and f_{X+Y} require X, Y to be independent.

Ex.

Let X, Y be independent continuous uniform r.v.s on $[a, b]$, i.e.

$$f_X = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $f_Y(y)$. Let

$$Z = X + Y .$$

Then

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) \, f_Y(z-x) \, dx$$

The integrand is $\left(\frac{1}{b-a}\right)^2$ iff both

$$a \le x \le b \quad \text{and} \quad a \le z-x \le b$$

i.e

$$a \le x \le b \quad \text{and} \quad z-b \le x \le z-a$$

(and 0 otherwise). This is equiv. to

$$\max(a, z-b) \le x \le \min(b, z-a)$$

Thus

$$f_Z(z) = \int_{\max(a,z-b)}^{\min(b,z-a)} \frac{1}{(b-a)^2}\,dz$$

$$= \frac{\min(b,z-a) - \max(a,z-b)}{(b-a)^2}$$

The interval of integration will be empty if either

$$z - a < a \iff z < 2a$$

or

$$z - b > b \iff z > 2b$$

Thus we require

$$2a \le z \le 2b$$

and

$$f_Z(z) = \begin{cases} \dfrac{\min(b, z-a) - \max(a, z-b)}{(b-a)^2} & \text{if } 2a \le z \le 2b \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: $Z = X + Y$ is <u>not</u> uniform.

<u>Ex</u>

Let X, Y be <u>independent</u> normal r.v.'s with

means: μ_X, μ_Y respectively

variances: σ_X^2, σ_Y^2 resp.

Thus

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot e^{-(x-\mu_X)^2/2\sigma_X^2}$$

and similarly for Y. Let

$$Z = X + Y \ . \ .$$

Then

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} \cdot e^{\frac{-(z-x-\mu_Y)^2}{2\sigma_Y^2}} \, dx$$

$$\vdots$$

(exercise)

$\vdots$

$$= \frac{1}{\sqrt{2\pi} \cdot \sqrt{\sigma_X^2 + \sigma_Y^2}} \cdot e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}}$$

Thus $Z = X + Y$ is <u>normal</u> with

mean: $\mu_X + \mu_Y$

variance: $\sigma_X^2 + \sigma_Y^2$.

It follows that if X, Y are independent normal r.v.s, then

$$aX + bY$$

is also normal.