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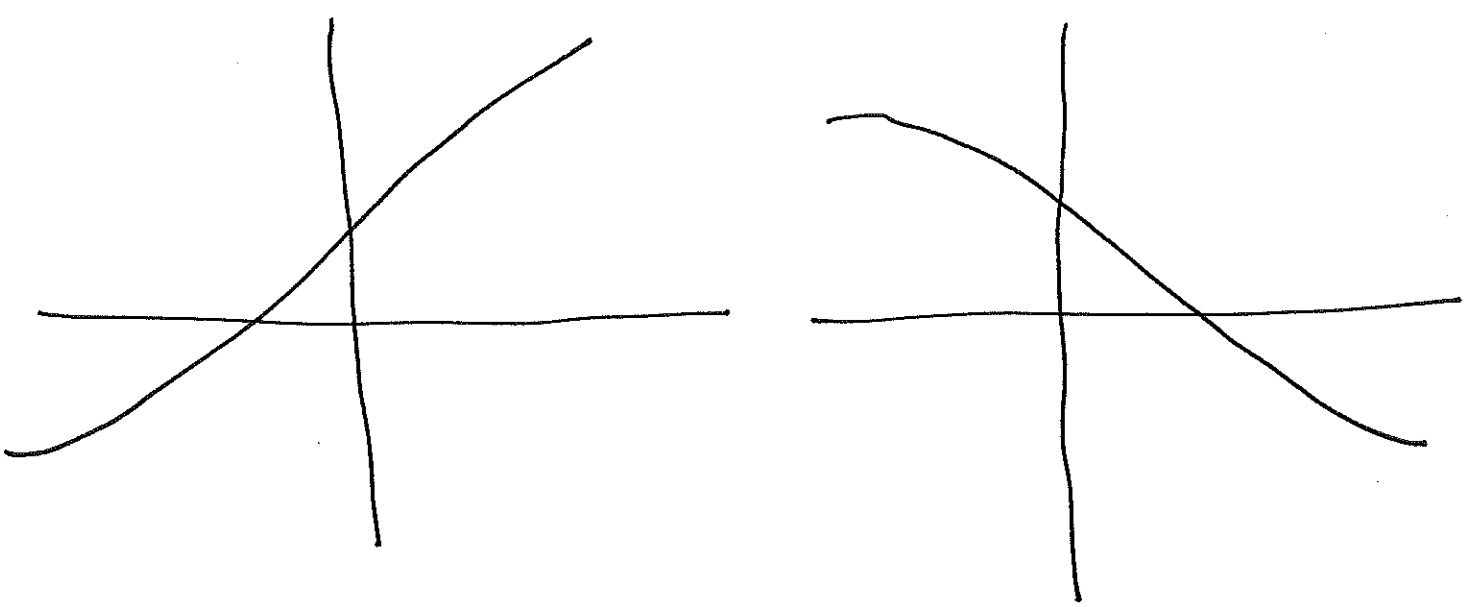
CSE 107 2-21-25

- Mid 2: Friday 2/28

Continuing ....

increasing: decreasing



Such functions are necessarily bijective, hence invertible.

Recall : $g : \mathbb{R} \to \mathbb{R}$

g is invertible iff there exists a fcn $h : \mathbb{R} \to \mathbb{R}$, $(h = g^{-1})$ s.t.

$$h(g(x)) = x \quad \text{for all } x \in \mathbb{R}$$

and

$$g(h(y)) = y \quad \text{" " } y \in \mathbb{R}.$$

We can show

(1) g mono. inc. $\Leftrightarrow$ h is mono. inc.

and

(2) g " dec. $\Leftrightarrow$ h is " dec.

<u>note</u>: if g is differentiable, then so is h, and

(1) iff $g' > 0$ iff $h' > 0$

and

(2) iff $g' < 0$ iff $h' < 0$

<u>Theorem</u>
Let X be a continuous r.v. and $Y = g(X)$. Then

$$\boxed{f_Y(y) = f_X(h(y)) \cdot |h'(y)|}$$

Proof

suppose g is monotonic increasing. Then so is h, and

(1) $$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \le h(y)) \quad \left\{ \begin{array}{l} \text{using that} \\ h \text{ is inc.} \end{array} \right.$$
$$= F_X(h(y))$$

(2) $$f_Y(y) = f_X(h(y)) \cdot h'(y) \underset{\uparrow}{=} f_X(h(y)) \, |h'(y)|$$
since $h' > 0$.

Exercise: do case g is decreasing.

<u>Ex.</u>

Let $Y = X^2$ where X is a cont. r.v. with range$(X) \subseteq [0,\infty)$.

so also range$(Y) \subseteq [0,\infty)$, and

$g : [0,\infty) \to [0,\infty)$, $g(x) = x^2$ is invertible, with $h(y) = g^{-1}(y) = \sqrt{y}$.

Thus

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} .$$

Two random variables : $g(X,Y)$

Ex.

let X, Y be independent cont. r.v.s, both uniform on $[0,1]$.

so

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

and similarly for Y.

Define Z by

$$Z = \max(X, Y)$$

Then for $0 \le z \le 1$ we have

$$F_Z(z) = P(Z \le z)$$

$$= P(\max(X, Y) \le z)$$

$$= P(X \le z, Y \le z)$$

$$= P(X \le z) \cdot P(Y \le z) \quad \left\{ \begin{array}{l} \text{by} \\ \text{indep.} \end{array} \right.$$

$$= F_X(z) \cdot F_Y(z)$$

$$= z \cdot z = z^2$$

i.e.

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ z^2 & 0 \le z \le 1 \\ 1 & z > 1 \end{cases}$$

hence

$$f_Z(z) = \begin{cases} 2z & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$\blacksquare$

<u>Exercise</u>

- do for X, Y both cont. unif. on $[a, b]$

- do this for X cont. unif. on $[a,b]$, and Y cont. unif on $[c,d]$.

<u>Note</u>

When $b<c$, then $[a,b]\cap[c,d]=\phi$, and $Y\geq X$, so $Z=\max(X,Y)=Y$, and $f_Z=f_Y$. Also, when $d<a$, we have $f_Z=f_X$.

So Assume both $c\leq b$ and $a\leq d$, so that $[a,b]\cap[c,d]\neq\phi$.

Sums and Convolutions

Let X, Y be indep. discrete r.v.'s and let

$$Z = X + Y$$

Then

$$p_Z(z) = P(Z = z) = P(X + Y = z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X = x, Y = y)$$

$$= \sum_{\{(x, z-x) \mid x \in \mathrm{range}(X)\}} P(X = x, Y = z - x)$$

$$= \sum_x P(X=x)\cdot P(Y=z-x) \quad \left\{ \begin{array}{l}\text{by indep.}\\ \text{of } X, Y\end{array}\right.$$

$$= \sum_x p_X(x)\cdot p_Y(z-x)$$

We call this the <u>convolution</u> <u>product</u>: $(p_X * p_Y)(z)$. Thus

$$p_{X+Y} = p_X * p_Y = p_Y * p_X.$$

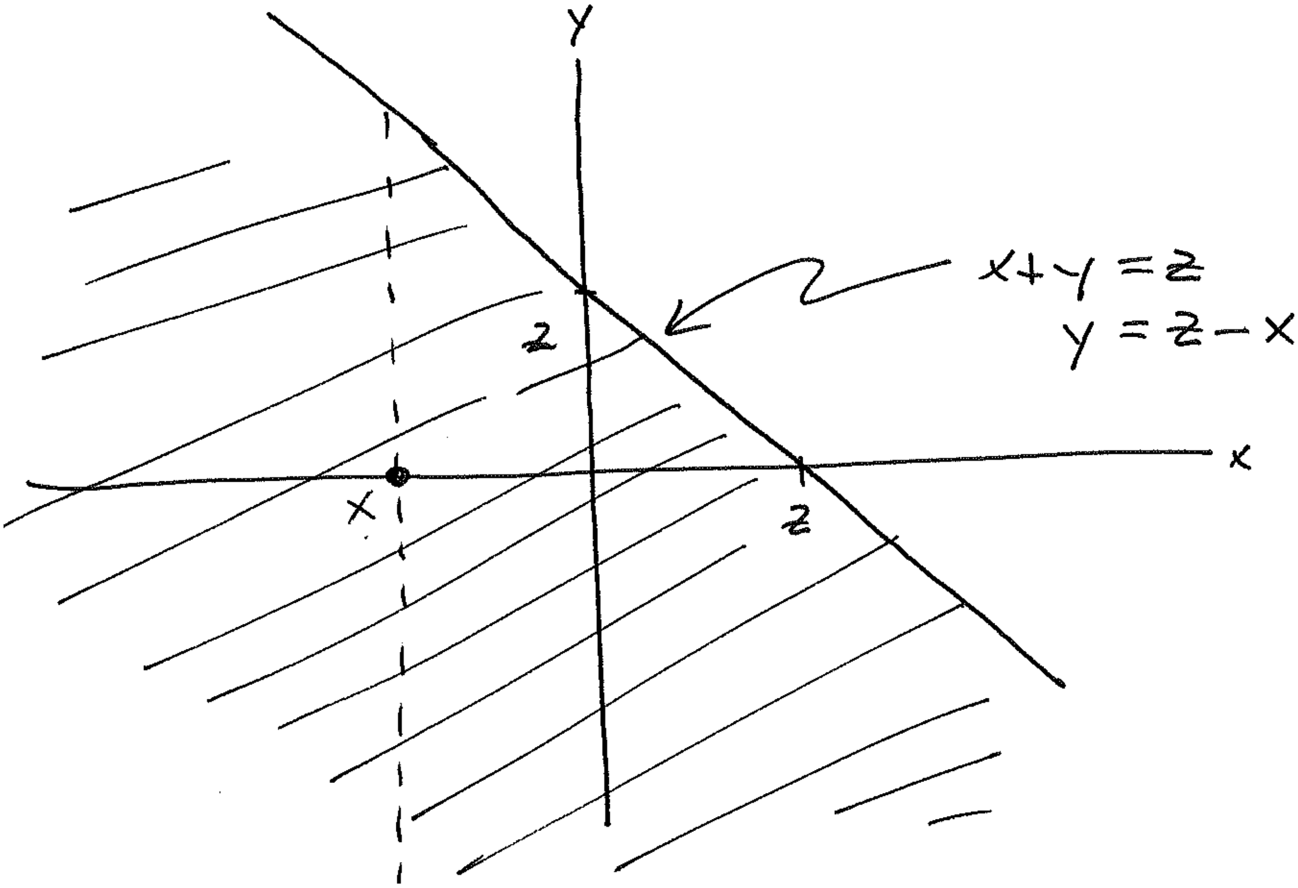
Now let X, Y be indep. cont. r.v.s, and

$$Z = X + Y.$$

Then

$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \iint_{\{(x,y) \mid x+y \le z\}} f_{X,Y}(x,y)\, dy\, dx$$

$y \le z - x$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) \cdot f_Y(y)\, dy\, dx \quad \Big\} \text{ by indep.}$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot \left(\int_{-\infty}^{z-x} f_Y(y)\, dy \right) dx$$