CSE 107 2-19-25

continuing - - -

a.) $f_Y(y) = \frac{1}{2}\left(\Phi(3-y) - \Phi(1-y)\right)$

b.)

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & \text{if } 1 \leq x \leq 3,\ y \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases}$$

c.) suppose $Y=3$. find prob. that $X \le 2$.

$$P(X \le 2 \mid Y=3) = \int_1^2 f_{X|Y}(x|3)\,dx$$

$$= \int_1^2 \frac{\frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}\,dx}{\Phi(0) - \Phi(-2)}$$

$$= \frac{\int_1^2 \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}\,dx}{\Phi(0) - \Phi(-2)}$$

$u = x-3$
$du = dx$
$x=1 \Rightarrow u=-2$
$x=2 \Rightarrow u=-1$

$$= \frac{\displaystyle\int_{-2}^{-1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du}{\Phi(0) - \Phi(-2)}$$

$$= \frac{\Phi(-1) - \Phi(-2)}{\Phi(0) - \Phi(-2)}$$

$$= \boxed{.2848}$$

d.) find $E[Y]$.

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} y \, \frac{1}{2} \int_{1}^{3} \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}} \, dx \, dy$$

$$= \frac{1}{2} \int_{1}^{3} \left[\int_{-\infty}^{\infty} y \cdot \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}} \, dy \right] dx$$

$$= \frac{1}{2} \int_{1}^{3} x \, dx = \frac{1}{2} \cdot \frac{1}{2} x^2 \Big|_{1}^{3}$$

$$= \frac{1}{4} (9-1) = \frac{8}{4} = \boxed{2}$$

4.1 Derived Distributions

If X is a r.v. and $Y = g(X)$, how can we compute PDF of Y

$$f_Y(y),$$

in terms of $f_X(x)$. We call $f_Y(y)$ a <u>derived distribution</u>.

The 2-step Process

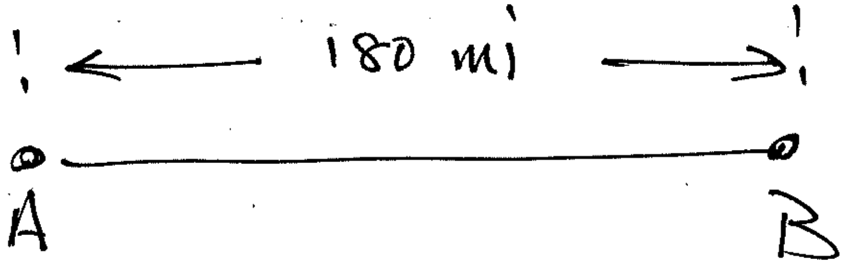
① find CDF of Y

$$F_Y(y) = P(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\, dx$$

(2) differentiate

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Ex.
A car travels 180 miles at a constant rate R. R is itself a r.v. which is uniform on $30 \le r \le 60$ (m.p.h.)



Let T be the time of travel. find $f_T(t)$

Solution

from $180 = R \cdot T$, we get

$$T = \frac{180}{R}$$

① $F_T(t) = P(T \le t)$

$$= P\left(\frac{180}{R} \le t\right)$$

$$= P\left(R \ge \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

② diff. w.r.t t:

$$f_T(t) = -f_R\left(\frac{180}{t}\right) \cdot \left(-\frac{180}{t^2}\right)$$

$$= \frac{180}{t^2} \cdot f_R\left(\frac{180}{t}\right)$$

We are given

$$f_R(r) = \begin{cases} \frac{1}{30} & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$f_T(t) = \begin{cases} \frac{180}{t^2} \cdot \frac{1}{30} & 30 \le \frac{180}{t} \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f_T(t) = \begin{cases} \dfrac{6}{t^2} & \text{if } \dfrac{1}{60} \le \dfrac{t}{180} \le \dfrac{1}{30} \\ 0 & \text{———} \end{cases}$$

$$= \begin{cases} \dfrac{6}{t^2} & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$\blacksquare$

Ex Let $Y = aX + b$. show

$$\boxed{f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)} \quad a \neq 0$$

Proof

① $F_Y(y) = P(Y \le y)$

$= P(aX + b \le y)$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ P\left(X \ge \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ 1 - P\left(X < \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

② diff. w.r.t. y

$$f_Y(y) = \begin{cases} \frac{1}{a} \cdot f_X\left(\frac{y-b}{a}\right) & a > 0 \\ -\frac{1}{a} \cdot f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$\therefore \quad f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right).$$

<u>Defn</u>

a fcn. $g : \mathbb{R} \to \mathbb{R}$ is <u>strictly monotonic</u> iff for all $x_1, x_2 \in \mathbb{R}$

(1) $x_1 < x_2 \implies g(x_1) < g(x_2)$ (increasing)

or

(2) $x_1 < x_2 \implies g(x_1) > g(x_2)$ (decreasing)