CSE 107 2-17-25

Supplemental Lecture

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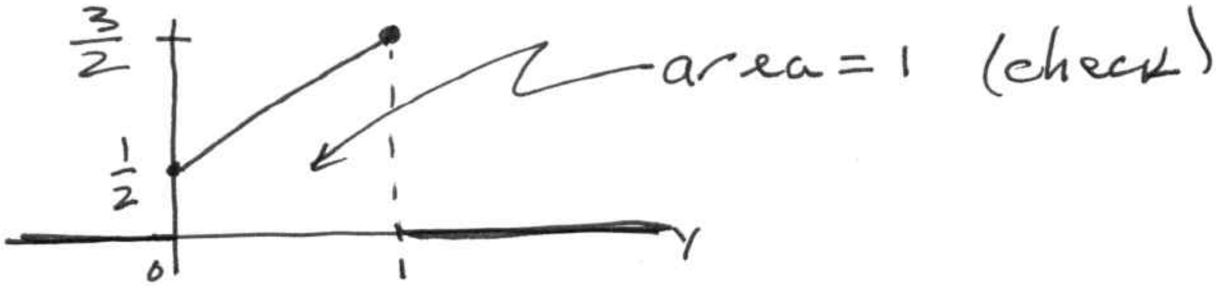
Ex.

let $X, Y$ be jointly continuous, with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$




and

$$f_{X|Y}(x|y) = \begin{cases} \frac{x+y}{y+\frac{1}{2}} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[X]$.

soln.

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx$$

$$= \int_0^1 x\left(\frac{x+y}{y+\frac{1}{2}}\right)dx = \frac{1}{y+\frac{1}{2}}\int_0^1 (x^2+xy)\,dx$$

$$= \frac{1}{y+\frac{1}{2}}\left(\tfrac{1}{3}x^3+\tfrac{1}{2}x^2y\right)\Big|_0^1$$

$$= \frac{\frac{1}{2}y+\frac{1}{3}}{y+\frac{1}{2}} \qquad (0 \le y \le 1)$$

and

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\,dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y+\frac{1}{3}}{\cancel{y+\frac{1}{2}}}\right)\cdot \cancel{\left(y+\tfrac{1}{2}\right)}\,dy$$

$$= \frac{1}{2}\frac{1}{2}y^2+\frac{1}{3}y\Big|_0^1 = \frac{1}{4}+\frac{1}{3} = \boxed{\frac{7}{12}}$$

another way:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$$

$$= \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy = xy + \frac{1}{2}y^2 \Big|_0^1$$

$$= \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^1 x\left(x+\tfrac{1}{2}\right)dx = \int_0^1 \left(x^2+\tfrac{1}{2}x\right) dx$$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{4}x^2 \Big|_0^1$$

$$= \tfrac{1}{3} + \tfrac{1}{4} = \boxed{\tfrac{7}{12}}$$

Back up a little, note $E[X]$ is a number, but

$$E[X \mid Y=y] = \begin{cases} \dfrac{\frac{1}{2}y + \frac{1}{3}}{y + \frac{1}{2}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a function of $y$.

we can compose this fcn. with a r.v. to get another r.v.

$$g(y) = E[X \mid Y = y]$$

compose with $Y$ to get

$$g(Y) = E[X \mid Y] = \frac{\frac{1}{2}Y + \frac{1}{3}}{Y + \frac{1}{2}}$$

we did this already:

$$E[E[X \mid Y]] = \frac{7}{12}$$

In general

$$\boxed{E[E[X \mid Y]] = E[X]}$$

this is part (2) of total expectation theorem, called <u>law of iterated expectations</u>.

## Independence

Let $X, Y$ be jointly continuous r.v.s

**Defn**

we say $X$ and $Y$ are independent iff

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

for all $x \in \text{range}(X)$, $y \in \text{range}(Y)$.

or

$$f_{X|Y}(x|y) = f_X(x) \quad \text{for } y \text{ s.t. } f_Y(y) > 0$$

or

$$f_{Y|X}(y|x) = f_Y(y) \quad \text{for } x \text{ s.t. } f_X(x) > 0.$$

Generalize to 3 or more r.v.s

$$\boxed{f_{X,Y,Z}(x,y,z) = f_X(x) \cdot f_Y(y) \cdot f_Z(z)}.$$

If $X, Y$ are independent, then any events of the form

$$\{X \in I\} \text{ and } \{Y \in J\}$$

would be independent. In Particular If $I = (-\infty, x]$, and $J = (-\infty, y]$ then

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\,ds\,dt$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_X(s) \cdot f_Y(t)\,ds\,dt$$

$$= \int_{-\infty}^{x} f_X(s)\,ds \cdot \int_{-\infty}^{y} f_Y(t)\,dt$$

$$= P(X \le x) \cdot P(Y \le y)$$

$$= F_X(x) \cdot F_Y(y) \,.$$

Note: Converse is also true, i.e. this eqn. $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$ implies independence. (exercise: Prove this.)

As in discrete case, we have for independent continuous r.v.s

(1) $E[X \cdot Y] = E[X] \cdot E[Y]$.

(2) $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$

and

(3) $Var(X+Y) = Var(X) + Var(Y)$.

Exercises

- Prove (1)
- Prove $X, Y$ indep. $\Rightarrow g(X), h(Y)$ indep.
  adapt proof in Prob. #44 (ch 2, p. 134)
- Prove (3). mimic proof from discrete case.

# 3.6 Bayes Rule

Recall Baye's Rule

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

If $A_1, A_2, \ldots, A_n$ form a Partition of $\Omega$, we may use the total Prob. thm

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

so for any $k$ in $1 \leq k \leq n$:

$$\boxed{P(A_k|B) = \frac{P(B|A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}}$$

If $X, Y$ are discrete r.v.,
this becomes

$$\boxed{p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\, p_X(x)}{\sum_t p_{Y|X}(y|t) \cdot p_X(t)}}$$

In the continuous case, we have

$$\boxed{f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) \cdot f_X(t)\, dt}}$$

Ex Let Y be a normal r.v. with s.d. 1 and mean X, another r.v. Suppose X is continuous uniform on [1, 3].

a) find $f_Y(y)$

Soln
We're given

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} & \text{if } 1 \le x \le 3,\ y \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\infty}^{\infty} f_{Y|X}(y|x) \cdot f_X(x)\,dx$$

$$= \frac{1}{2}\int_{1}^{3} \frac{1}{\sqrt{2\pi}} \cdot e^{-(x-y)^2/2}\,dx$$

$\underline{\underline{\text{u-sub.}}}$ $u = x - y$ $\quad x=1 \Rightarrow u = 1-y$
$du = dx$ $\quad x=3 \Rightarrow u = 3-y$

$$= \frac{1}{2}\int_{1-y}^{3-y} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

$$= \frac{1}{2}\left(\Phi(3-y) - \Phi(1-y)\right)$$

b.) find $f_{X|Y}(x|y)$

soln

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)}$$

Exercises:

c.) suppose we sample $Y$ and get $Y=3$. What is the that $X \leq 2$, i.e. $P(X \leq 2)$?

d.) find $E[Y]$.