CSE 107 2-15-24

Supplemental Lecture

Ex.

Let Y be geometric with parameter p.

$$\overbrace{\underbrace{0\ 0\ 0\ 0\ 0\ 0 \cdots 0}_{m}\ \underbrace{0\ 0 \cdots\cdots 0\ 1}_{X}}^{Y}$$

so

$Y = \#$ flips to 1^{st} head

$X = \#$ flips after m^{th} to 1^{st} head

let

$$A = \{Y > m\}$$

We must show: $P_Y(k) = P_{X|A}(k) \quad (k = 1, 2, 3, \ldots)$

We know: $P_Y(k) = p(1-p)^{k-1} \quad (k = 1, 2, 3, \ldots)$

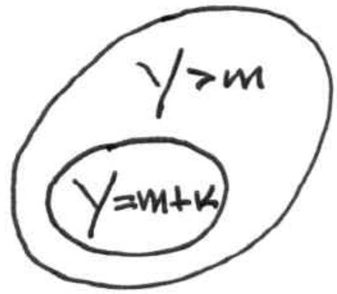
Also (for $k \geq 1$)

$$p_{X|A}(k) = P(X = k \mid A)$$

$$= P(Y - m = k \mid Y > m)$$

$$= \frac{P(Y = m+k,\ Y > m)}{P(Y > m)}$$

$$= \frac{P(Y = m+k)}{P(Y > m)}$$

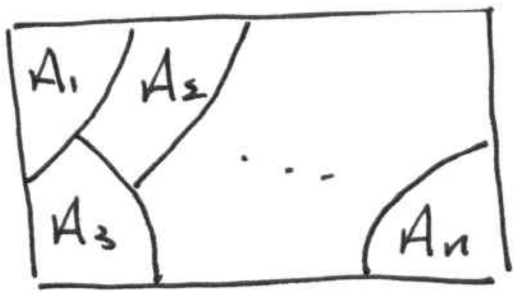


$$= \frac{p \cdot (1-p)^{m+k-1}}{(1-p)^m}$$

$$= \frac{p \cdot \cancel{(1-p)^m}\,(1-p)^{k-1}}{\cancel{(1-p)^m}}$$

$$= p \cdot (1-p)^{k-1} = p_Y(k) \quad (k = 1, 2, 3, \ldots).$$

If $A_1, A_2, \ldots, A_n \subseteq \Omega$ form a partition of Ω:



Ω

- $A_i \cap A_j = \phi$ for $i \neq j$
- $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$,

then the total Probability theorem gives

$$P(X \leq x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \leq x \mid A_i)$$

Thus

$$F_X(x) = \sum_{i=1}^{n} P(A_i) \cdot F_{X|A_i}(x)$$

diff. w.r.t. x to get

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)}$$

As in the discrete case we can condition X on another r.v. Y

Defn

If X, Y are jointly continuous r.v.s, and $f_Y(y) > 0$ for all $y \in \mathbb{R}$, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

observe

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

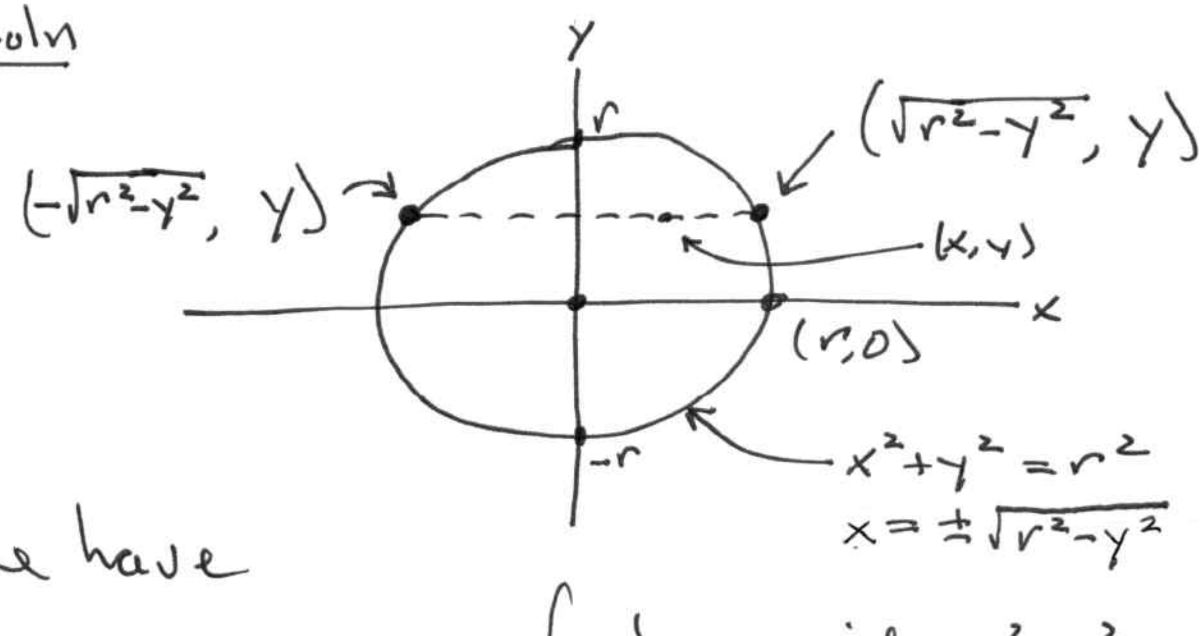
$$= \frac{f_Y(y)}{f_Y(y)} = 1 .$$

so $f_{X|Y}(x|y)$ is a valid PDF,

for any $y \in \mathbb{R}$ for which $f_Y(y) > 0$.

Ex throw a dart at a circular target, radius r, centered at $(0,0)$. Assume target is hit, but otherwise the impact point (X,Y) is uniformily distributed. find $f_{X,Y}(x,y)$, $f_Y(y)$ and $f_{X|Y}(x|y)$.

Soln



we have

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise}\end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \times \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}$$

$$= \frac{2\sqrt{r^2-y^2}}{\pi r^2}$$

$$= \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } |y| \le r \\ 0 & \text{otherwise} \end{cases}$$

Finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r \\ 0 & \text{otherwise} \end{cases}$$

$\blacksquare$

<u>Exercise</u>: find $f_X(x)$ and $f_{Y|X}(y|x)$.

All the same identities that hold for PMFs also hold for PDFs, such as

- $f_{X|Y}(x|y) \cdot f_Y(y) = f_{X,Y}(x,y)$
- $f_{Y|X}(y|x) \cdot f_X(x) = f_{X,Y}(x,y)$

and multiplication rule:

- $f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z) \cdot f_{Y|Z}(y|z) \cdot f_Z(z)$

Conditional Expectation

Let X, Y be jointly cont. r.v.s and $A \subseteq \Omega$ with $P(A) > 0$.

Defns

- $E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx$

- $E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$

Expected value rule

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx$

- $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\, dx$

Total expectation theorem

- if $A_1, A_2, \dots, A_n$ partition Ω, and $P(A_i) > 0$ for all i, then

(1) $E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$ ✓

- similarly

(2) $E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X|Y=y]\, dy$

Proof of (1)

start with total Prob. thm.

$$f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

multiply by x and integrate w.r.t. x from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{-\infty}^{\infty} x \cdot \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)\,dx$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \int_{-\infty}^{\infty} x \, f_{X|A_i}(x)\,dx$$

$$\therefore \; E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i] \quad \blacksquare$$

Proof of (2)

start with RHS

$$RHS = \int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right) \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = E[X] = LHS.$$

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