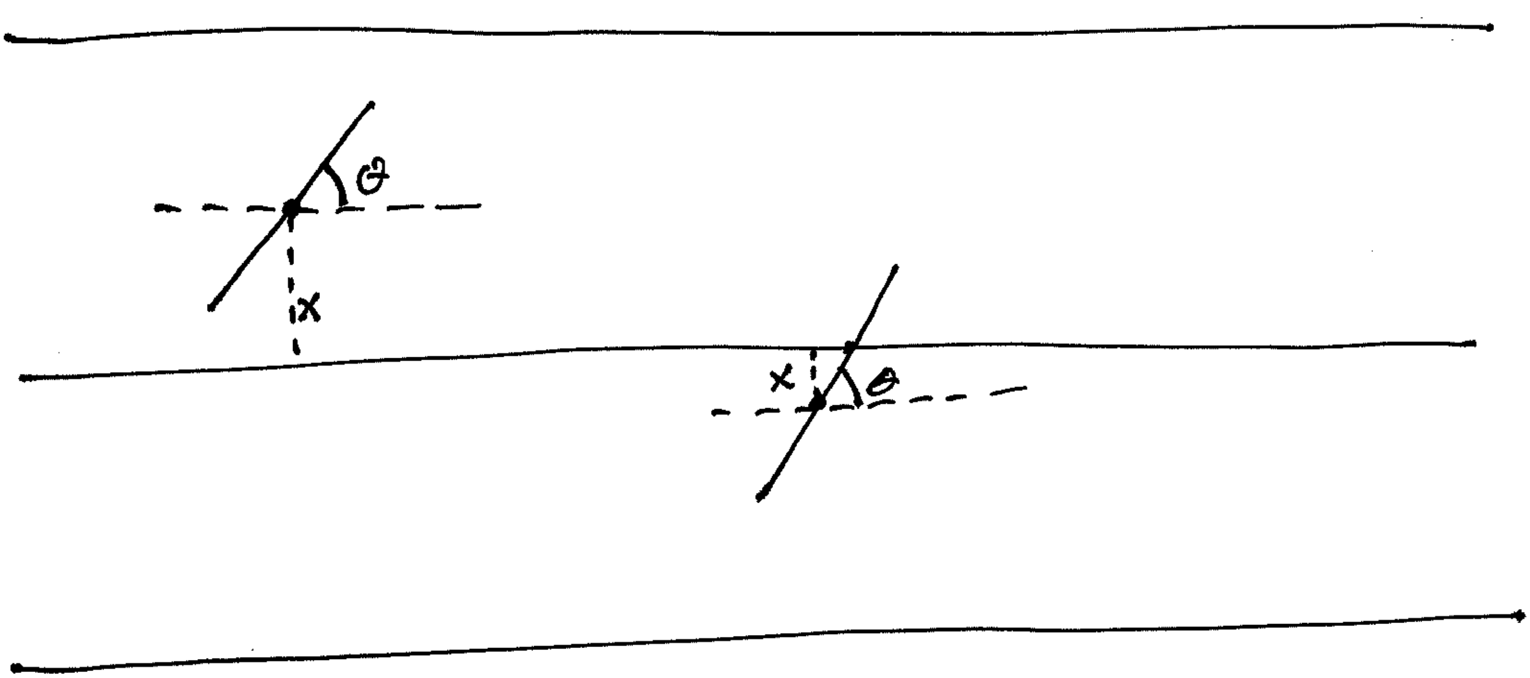
CSE 107 2-14-25

Continue .. Buffon's Needle

Let X be vertical from midpoint of needle to nearest line (so $0 \le X \le \frac{d}{2}$)
Let Θ be the acute angle made by the needle and the horizontal ($0 \le \Theta \le \frac{\pi}{2}$)

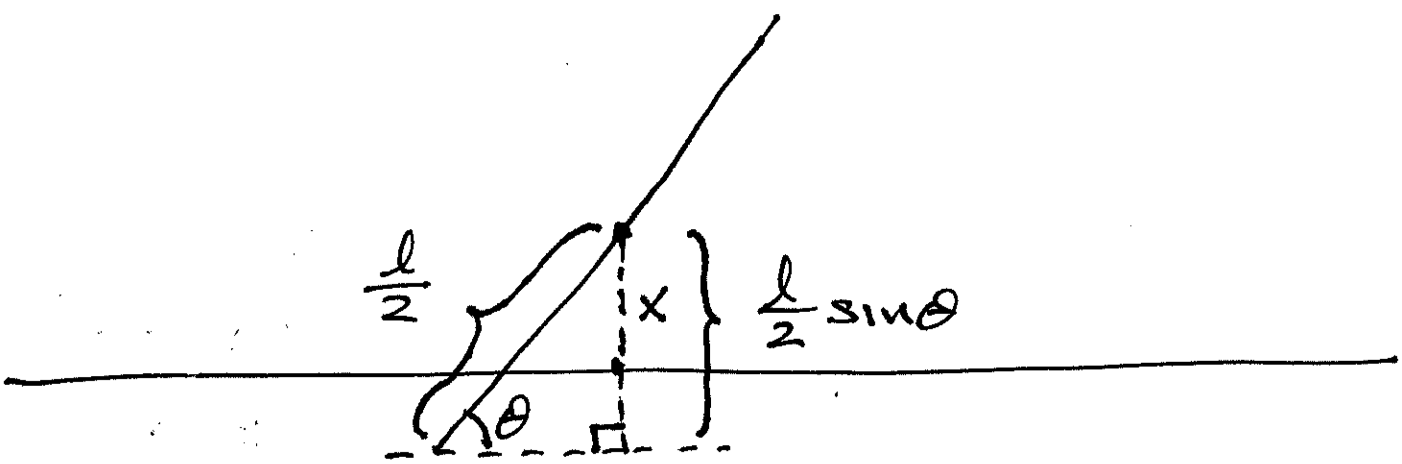
⋮



⋮

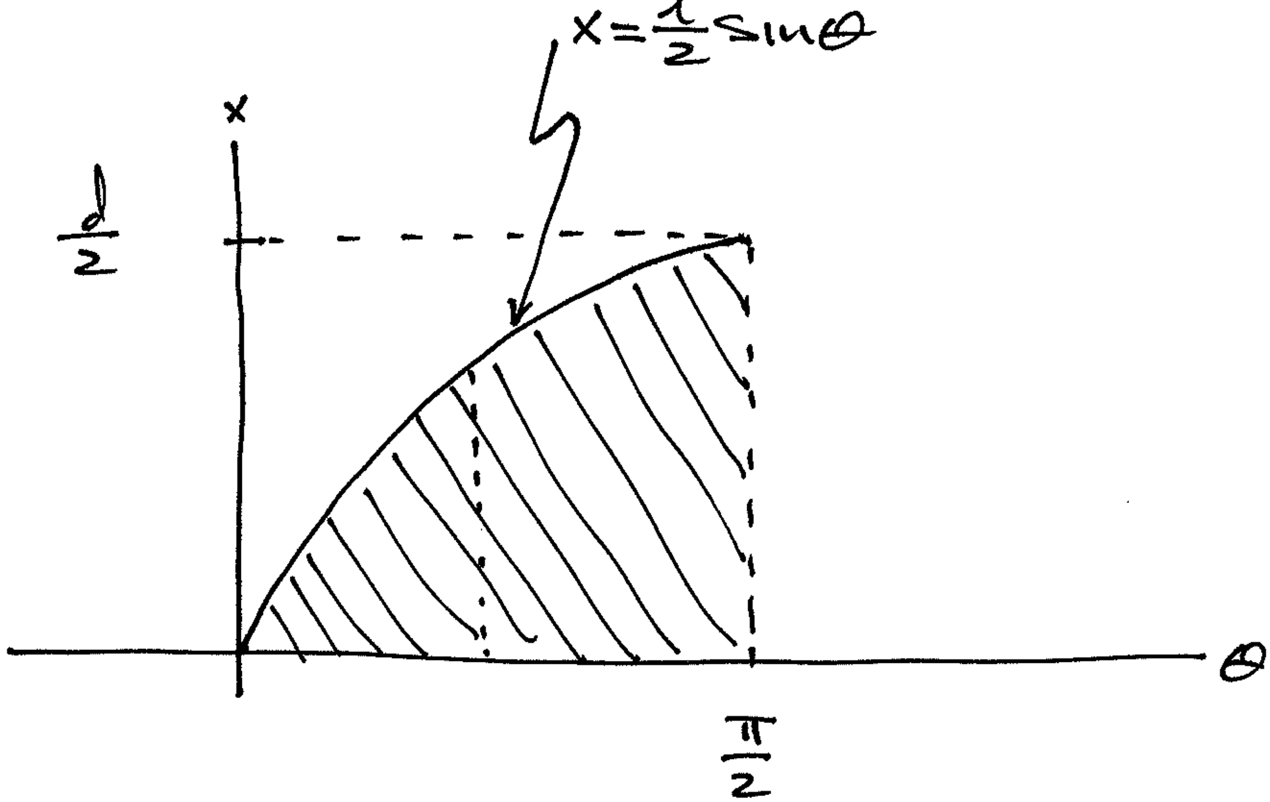
We model (X, Θ) by a uniform distribution on $[0, \frac{d}{2}] \times [0, \frac{\pi}{2}]$, with area $= \frac{\pi d}{4}$.

$$f_{X,\Theta}(x,\theta) = \begin{cases} \frac{4}{\pi d} & \text{if } 0 \le x \le \frac{d}{2} \text{ and} \\ & \quad 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$



The needle intersects a line iff

$$X \le \frac{l}{2}\sin\Theta$$



Thus

$$P\left(X \le \frac{l}{2}\sin\theta\right) = \iint\limits_{x \le \frac{l}{2}\sin\theta} f_{X,\Theta}(x,\theta)\, dx\, d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \int_0^{l/2 \sin\theta} dx\, d\theta$$

$$= \frac{4}{\pi d} \cdot \int_0^{\pi/2} \frac{l}{2} \sin\theta\, d\theta$$

$$= \frac{2l}{\pi d}\left(-\cos\theta\right)\Big|_0^{\pi/2}$$

$$= -\frac{2l}{\pi d}(0-1) = \boxed{\frac{2l}{\pi d}}$$

If we pick $l = d$, then

$$P(\text{intersect}) = \frac{2}{\pi}$$

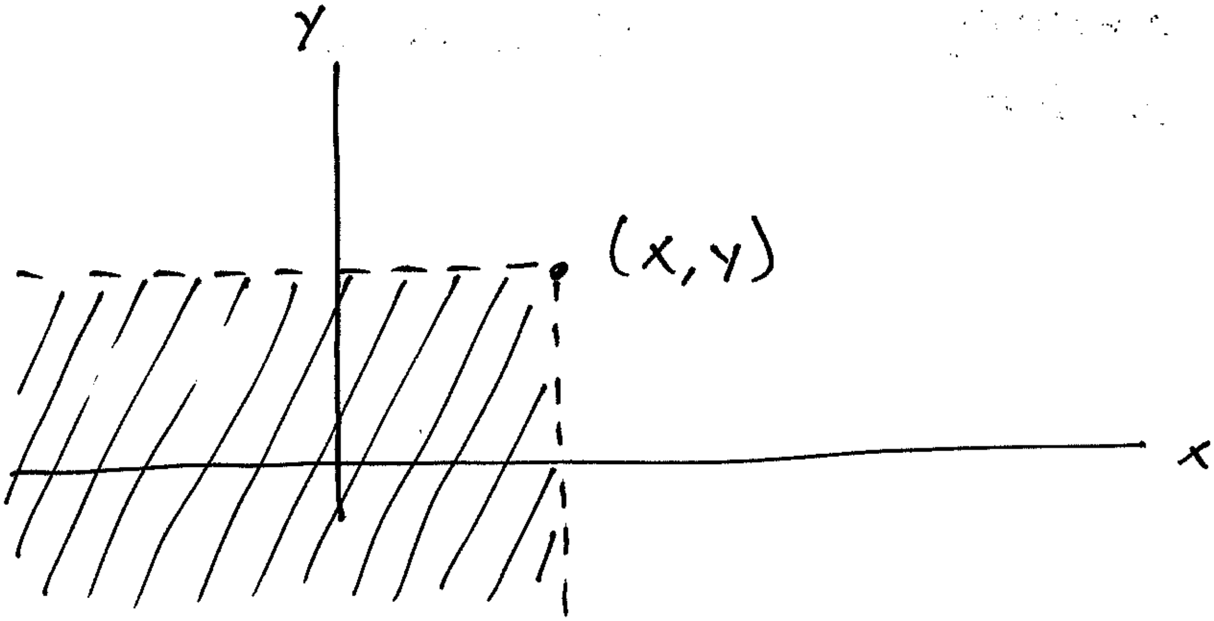
$$\pi = \frac{2}{P(\text{intersection})} \approx \frac{2}{\left(\frac{\#\text{intersect}}{\#\text{tosses}}\right)}$$

Joint CDFs :

If X, Y are two jointly cont. r.vs, their joint CDF is

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$



So

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y \partial x}(x,y)$$

<u>Expected value rule</u>

If $g: \mathbb{R}^2 \to \mathbb{R}$, then

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) \cdot f_{X,Y}(x,y)\, dx\, dy$$

Special case

$$E[aX+bY+c]$$

$$= \iint_{\mathbb{R}^2} (ax+by+c) f_{X,Y}(x,y)\,dx\,dy$$

$$= \text{.....(exercise).....}$$

$$= aE[X]+bE[Y]+c$$

We can re-do all of this for any number of r.v.s. !

$$E[a_1X_1+a_2X_2+\cdots+a_nX_n]=a_1E[X_1]+\cdots+a_nE[X_n]$$

3.5 Conditioning

Defn

Let $P(A) > 0$, and let X be a cont. r.v. The conditional PDF of X given A is a fcn. $f_{X|A}(x)$ satisfying

$$P(X \in I | A) = \int_I f_{X|A}(x)\, dx$$

for 'any' subset $I \subseteq \mathbb{R}$

We require

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$$

<u>Read</u>: special case $A = \{X \in J\}$ for some $J \subseteq \mathbb{R}$.

<u>EX</u>

The time T until a new light bulb burns out is exponential with parameter λ.

so

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Alice leaves the room and returns t_0 hours later, finding the light still on. So, the event

$$A = \{T > t_0\}$$

has occurred. Let X be the additional time until burn out.

i.e. $X = T - t_0$, $T = X + t_0$

Find the conditional PDF

$$f_{X|A}(x)$$

Solution

First find conditional CDF

$$F_{X|A}(x) = P(X \leq x \mid A),$$

then differentiate to get

$$f_{X|A}(x) = \frac{d}{dx} F_{X|A}(x)$$

Recall

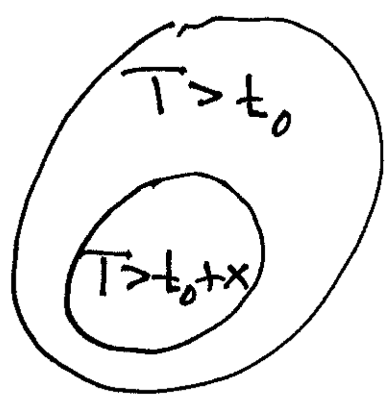
$$P(T>a)=\begin{cases} e^{-\lambda a} & \text{if } a>0 \\ 1 & \text{if } a\le 0 \end{cases}$$

Thus (for $x\ge 0$):

$$P(X>x\mid A)=P(T>t_0+x\mid T>t_0)$$

$$=\frac{P(T>t_0+x,\ T>t_0)}{P(T>t_0)}$$

$$=\frac{P(T>t_0+x)}{P(T>t_0)}$$



$$= \frac{e^{-\lambda(t_0+x)}}{e^{-\lambda t_0}}$$

$$= \frac{\cancel{e^{-\lambda t_0}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t_0}}} = e^{-\lambda x}$$

Thus

$$F_{X|A}(x) = P(X \le x | A)$$

$$= 1 - P(X > x | A)$$

$$= \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This is exponential with same λ, showing that the exponential r.v. is memoryless.