CSE 107 2-12-25

Recall

if X is normal with μ, σ (mean, stdv.), then so is $Y = aX + b$ $(a \neq 0)$.

Proof

assume $a > 0$. (leave $a < 0$ exercise.)

$$F_Y(y) = P(Y \leq y)$$

$$= P(aX + b \leq y)$$

$$= P\left(X \leq \frac{y-b}{a}\right)$$

$$= F_X\left(\frac{y-b}{a}\right)$$

differentiate w.r.t y to get

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a}$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{\frac{-\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,(a\sigma)}\cdot e^{\frac{-\left(\frac{y-b-a\mu}{a}\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,(a\sigma)}\cdot e^{-\frac{(y-(a\mu+b))^2}{2(a\sigma)^2}}$$

so Y is normal with mean $a\mu+b$ and std. dev. $a\sigma$, so variance $a^2\sigma^2$.

Ex.

What is the prob. that a normal r.v. X is within 1 standard dev. of its mean?

i.e. find

$$P(\mu-\sigma \le X \le \mu+\sigma)$$

$$= P\left(\frac{(\mu-\sigma)-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{(\mu+\sigma)-\mu}{\sigma}\right)$$

$$= P(-1 \le Y \le 1)$$

std. normal

$$= P(Y \le 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2(.8413) - 1 = \boxed{.6826}$$

- same question, but 2σ and 3σ

2σ: $2\Phi(2) - 1 = \ldots = \boxed{.9544}$

3σ: $2\Phi(3) - 1 = \cdots = \boxed{.9974}$

3.4 Joint PDFs of several r.v.s

Defn

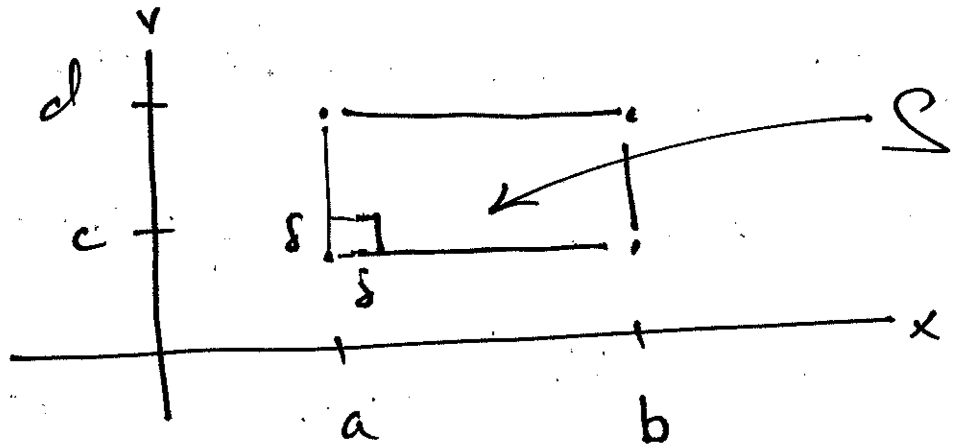
We say r.v.s X, Y are jointly continuous iff there exists a fcn. $f_{X,Y}(x,y)$ such that

$$P((X,Y)\in S) = \iint_S f_{X,Y}(x,y)\,dx\,dy$$

where $S \subseteq \mathbb{R}^2$. We call $f_{X,Y}(x,y)$ the joint PDF of X and Y.

In particular, if S is a rectangle

$$S = [a,b] \times [c,d]$$



Then

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\, dx\, dy$$

Note: if $S = \mathbb{R}^2$, then

$$\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = P(\Omega) = 1$$

If $\delta > 0$ is small, and $f_{X,Y}(x,y)$ is continuous near (a,c), then

$$P(a \le X \le a+\delta,\ c \le Y \le c+\delta) \quad \longleftarrow \text{mass}$$

$$= \int_a^{a+\delta} \int_c^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$$

$$\approx f_{X,Y}(a,c) \cdot \underbrace{\delta^2}_{\text{area}}$$

$\therefore f_{X,Y}(x,y)$ has units : mass per unit area, i.e. density.

Recall if $T \subseteq \mathbb{R}$, we have

$$P(X \in T) = \int_T f_X(x)\,dx$$

But also

$$P(X \in T) = P(X \in T,\ Y \in \mathbb{R})$$

$$= \iint_{T \times \mathbb{R}} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_T \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \right) dx .$$

Since this holds for 'any' $T \subseteq \mathbb{R}$, we have

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy}$$

also

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx}$$

observe

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \right] dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dy\,dx$$

and

$$E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy$$

Ex. Joint uniform continuous PDF

on $R = [a,b] \times [c,d]$

$$f_{X,Y}(x,y) = \begin{cases} \alpha & \text{if } \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

We have

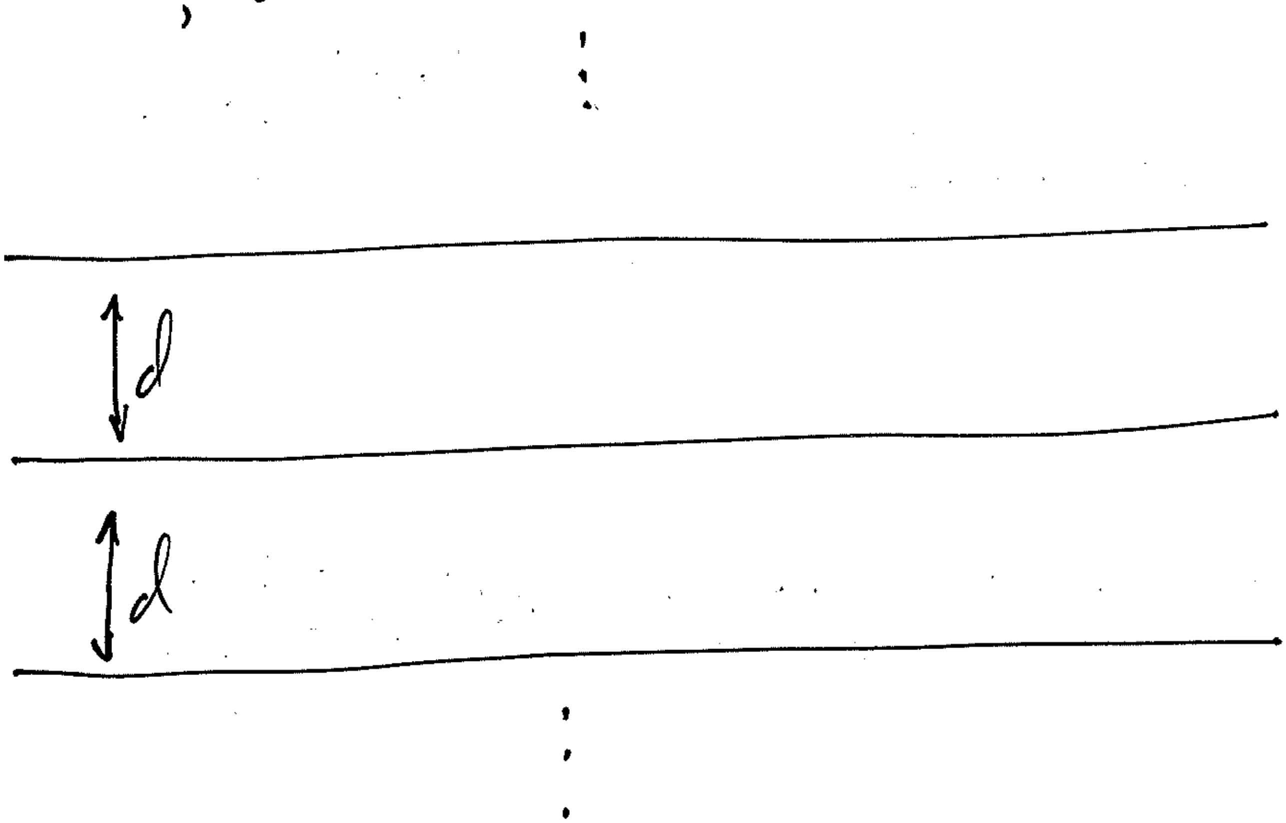
$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \iint_R \alpha\,dx\,dy$$

$$= \alpha \cdot \iint_R dx\,dy = \alpha \cdot \text{area}(R)$$

$$\therefore \quad \alpha = \frac{1}{\text{area}(R)} = \frac{1}{(b-a)(d-c)}$$

Ex. Buffon's needle.

A surface is ruled with parallel lines, distance d apart



a needle of length l is tossed on the surface. What is the Prob. that the needle intersect a line

(assume $l < d$, so at most 1 intersect.)