CSE 107 1-29-25

- midterm 1 : 1-31-25 Friday.
- hw3 : Problem 8 added.

let X be geometric with param p.

Let $A_1 = \{X = 1\} = \{1^{st}$ flip is head$\}$

$A_2 = \{X > 1\} = \{1^{st}$ flip is tail$\}$

so $P(A_1) = p$ and $P(A_2) = 1-p$.

<u>note</u> $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = \Omega$

Also $E[X \mid A_1] = 1$, and

$$E[X \mid A_2] = E[1 + X] = 1 + E[X]$$

Hence, by total expectation thm

$$\cancel{E[X]} = P(A_1) \cdot E[X \mid A_1] + P(A_2) \cdot E[X \mid A_2]$$

$$= p \cdot 1 + (1-p) \cdot (1 + E[X])$$

$$\cancel{E[X]} = \cancel{p} + (1 - \cancel{p}) + \cancel{E[X]} - p E[X]$$

$$\therefore \quad p E[X] = 1$$

$$\therefore \quad E[X] = \boxed{\frac{1}{p}}$$

Similarly

$$E[X^2 \mid A_1] = 1$$

and

$$E[X^2 \mid A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2E[X] + E[X^2]$$

hence

$$\cancel{E[X^2]} = P(A_1)\cdot E[X^2 \mid A_1] + P(A_2)\cdot E[X^2 \mid A_2]$$

$$= p \cdot 1 + (1-p)\left(1 + 2\cdot\tfrac{1}{p} + E[X^2]\right)$$

$$= \cancel{p} + (1-\cancel{p}) + (1-p)\tfrac{2}{p} + \cancel{E[X^2]} - pE[X^2]$$

$$pE[X^2] = 1 + \frac{2-2p}{p} = \frac{p+2-2p}{p} = \frac{2-p}{p}$$

$$\therefore \; E[X^2] = \frac{2-p}{p^2} = \frac{2}{p^2} - \frac{1}{p}$$ ✓

finally

$$Var(X) = \left(\frac{2}{p^2} - \frac{1}{p}\right) - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \boxed{\frac{1-p}{p^2}}$$ ✓

2.7 Independence

Defn

We say two r.v.s X, Y are <u>independent</u> iff for all $x \in \text{range}(X)$, $y \in \text{range}(Y)$ we have

$$\{X = x\}, \{Y = y\}$$

are independent events, i.e.

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

equivalently

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)}$$

This is equivalent to

$$\frac{P_{X,Y}(x,y)}{P_Y(y)} = P_X(x)$$

i.e.

$$\boxed{P_{X|Y}(x|y) = P_X(x)}$$

Defn

We say 2 r.v.s X, Y are conditionally indep. given event A iff

$$P(X=x, Y=y \mid A) = P(X=x \mid A) \cdot P(Y=y \mid A)$$

i.e.

$$\boxed{p_{X,Y|A}(x,y) = p_{X|A}(x) \cdot p_{Y|A}(y)}$$

Warning:

$$\boxed{\text{cond. indep}} \;\; \begin{array}{c} \not\Rightarrow \\ \not\Leftarrow \end{array} \;\; \boxed{\text{indep.}}$$

see fig. 2.15 on p.111

<u>Theorem</u>

If X, Y are indepenent r.v.s,

then

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

<u>Proof</u>

$$E[XY] = \sum_x \sum_y xy \, P_{X,Y}(x,y)$$

$$= \sum_x \sum_y xy \, P_X(x) \cdot P_Y(y)$$

$$= \left(\sum_x x P_X(x)\right) \cdot \left(\sum_y y P_Y(y)\right)$$

$$= E[X] \cdot E[Y]$$

∎

Thus

If X, Y are indep., then so are $g(X)$ and $h(Y)$.

see Problem #44, p. 134 for proof.

Thus

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)].$$

Theorem

if X, Y are indep., then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Proof

let X, Y be indep. define

$$U = X - E[X]$$

$$V = Y - E[Y]$$

so $$E[U^2] = \mathrm{Var}(X)$$

and $$E[V^2] = \mathrm{Var}(Y)$$

Also

$$E[U] = E[X] - E[X] = 0$$

$$E[V] = 0$$

Also

$$U + V = (X+Y) - E[X+Y]$$

Thus

$$\operatorname{Var}(X+Y) = E[(U+V)^2]$$

$$= E[U^2 + 2UV + V^2]$$

$$= E[U^2] + 2\cancel{\underbrace{E[UV]}_{\substack{E[U]\cdot E[V] \\ 0 \ \cdot \ 0}}} + E[V^2]$$

$$= Var(X) + Var(Y).$$

$\blacksquare$