CSE 107 1-27-25 

Recall:

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

and

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)}$$

Ex.

A professor answers questions wrongly with prob. $\frac{1}{4}$, indep. of any other questions. The # of questions asked is 0, 1, or 2 with equal probability $\frac{1}{3}$. Let

$X$ = # questions asked

$Y$ = # questions answered wrongly.

We have

$$P_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Also

#asked #wrong

$$P_{Y|X}(y|x) = \begin{cases} 1 & x=0,\ y=0 \\ 3/4 & x=1,\ y=0 \\ 1/4 & x=1,\ y=1 \\ 9/16 & x=2,\ y=0 \\ 3/8 & x=2,\ y=1 \\ 1/16 & x=2,\ y=2 \\ 0 & \text{otherwise} \end{cases}$$

Table form

$P_X(x)$

| | | |
|---|---|---|
| 1/3 | 1/3 | 1/3 |
| 0 | 1 | 2 |

$\underbrace{\qquad}_{x}$

$P_{Y|X}(y|x)$

| y \ x | 0 | 1 | 2 |
|---|---|---|---|
| 2 | | | 1/16 |
| 1 | | 1/4 | 3/8 |
| 0 | 1 | 3/4 | 9/16 |

$\underbrace{0 \quad 1 \quad 2}_{x}$

Find $P_{X,Y}(x,y)$

| $y \backslash x$ | 0 | 1 | 2 |
|---|---|---|---|
| 2 | | | $1/48$ |
| 1 | | $1/12$ | $1/8$ |
| 0 | $1/3$ | $1/4$ | $3/16$ |

| $y$ | $P_Y(y)$ |
|---|---|
| 2 | $1/48$ |
| 1 | $10/48$ |
| 0 | $37/48$ |

using: $P_{X|Y}(x|y) = \dfrac{P_{X,Y}(x,y)}{P_Y(y)}$

| $y \backslash x$ | 0 | 1 | 2 |
|---|---|---|---|
| 2 | | | 1 |
| 1 | | $2/5$ | $3/5$ |
| 0 | $16/37$ | $12/37$ | $9/37$ |

# Conditional Expectation

let $X$ be a random var. and $A \subseteq \Omega$ an event, $P(A) > 0$.

$$\boxed{E[X|A] = \sum_x x\, P_{X|A}(x)}$$

also

$$\boxed{E[g(X)|A] = \sum_x g(x) \cdot P_{X|A}(x)}$$

can condition on $A = \{Y = y\}$

$$\boxed{E[X|Y=y] = \sum_x x\, P_{X|Y}(x|y)}$$

Theorem

Total Expectation Theorem:

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

where $A_1, A_2, \ldots, A_n$ form a partition of $\Omega$.

Proof

By the total prob. thm., we have

$$P(X=x) = \sum_{i=1}^{n} P(A_i) \cdot P(X=x \mid A_i)$$

so

$$P_X(x) = \sum_{i=1}^{n} P(A_i) \cdot P_{X|A_i}(x)$$

multiply both sides by x, and sum over x to get

$$E[X] = \sum_x x \, P_X(x)$$

$$= \sum_x x \cdot \sum_{i=1}^{n} P(A_i) \cdot P_{X|A_i}(x)$$

$$= \sum_x \sum_{i=1}^{n} P(A_i) \cdot x \, P_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \sum_x x \, P_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i)\, E[X \mid A_i]$$

$\blacksquare$

see similar identities on P. 104.

## Mean and Variance of Geometric

Recall, the geometric r.v. with parameter $p$ has PMF

$$p_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then

(1) $E[X] = \sum_{K=1}^{\infty} K (1-p)^{K-1} \cdot p$

(2) $E[X^2] = \sum_{K=1}^{\infty} K^2 (1-p)^{K-1} \cdot p$

and

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

Exercise: show

(1) $E[X] = \frac{1}{p}$

(2) $E[X^2] = \frac{2-p}{p^2}$