Ex. the "hat Problem"

n People throw their hats into a box, then each picks one hat at random. Thus

$$\{\text{hats}\} \longrightarrow \{\text{People}\}$$

is a random mapping.

$$P(\text{get own hat back}) = \frac{1}{n}.$$

let X = # of people who get their own hat.

Find $E[X]$.

observe $X = X_1 + X_2 + \cdots + X_n$

where each X_i is Bernoulli

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person gets own hat} \\ 0 & \text{if not} \end{cases}$$

and

$$P(X_i = 1) = \frac{1}{n}$$

so

$$E[X] = E[X_1 + X_2 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= \underbrace{p + p + \cdots + p}_{n}$$

$$= n \cdot p = n \cdot \frac{1}{n} = \boxed{1}$$

2.6 Conditioning

Recall

$$p_X(x) = P(X = x)$$

This works for any valid prob. law $P(\cdot)$, including conditional probability laws.

<u>Defn</u>
let X be a r.v., $A \subseteq \Omega$ with $P(A) > 0$. The <u>conditional PMF</u>

$p_{X|A}(x)$ is

$$p_{X|A}(x) = P(\{X=x\} \mid A)$$

$$= \frac{P(\{X=x\} \cap A)}{P(A)}$$

note: $A = \bigcup_{x \in \text{range}(X)} (\{X=x\} \cap A)$

is a disjoint union

so by total Prob. thm

$$P(A) = \sum_x P(\{X=x\} \cap A)$$

Thus

$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\}\cap A)}{P(A)}$$

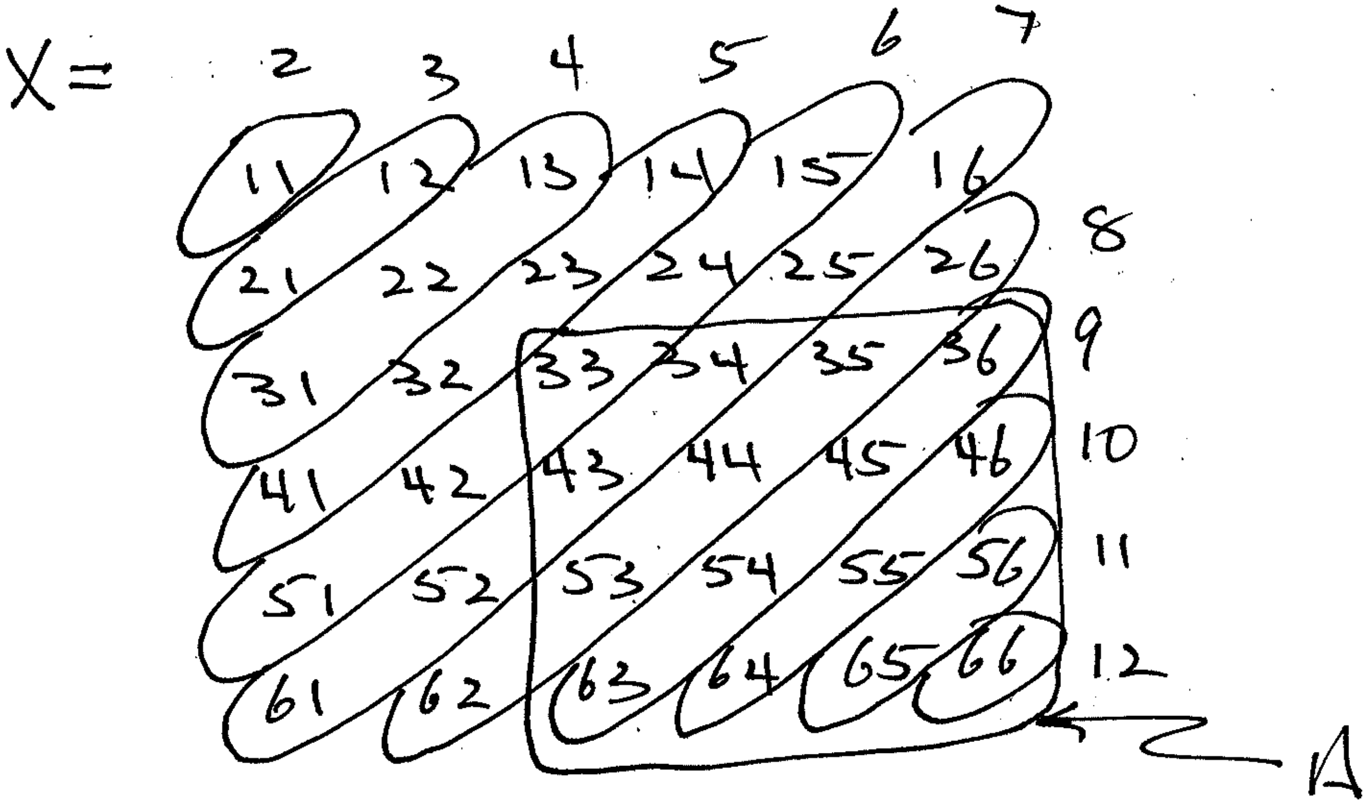
$$= \frac{\sum_x P(\{X=x\}\cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)} = 1 \checkmark$$

so $P_{X|A}(x)$ is a valid PMF.

<u>Ex</u> roll 2 indep., fair, 6-sided dice, let X be sum.

$$p_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & k = 3, 11 \\ 3/36 & k = 4, 10 \\ 4/36 & k = 5, 9 \\ 5/36 & k = 6, 8 \\ 6/36 & k = 7 \\ 0 & \text{otherwise} \end{cases}$$



let $A = \{\min \geq 3\}$, $P(A) = \frac{16}{36}$

$$P_{X|A}(k) = \begin{cases} 1/16 & \text{if } k = 6, 12 \\ 2/16 & \text{if } k = 7, 11 \\ 3/16 & \text{if } k = 8, 10 \\ 4/16 & \text{if } k = 9 \\ 0 & \text{otherwise} \end{cases}$$

Ex flip a weighted coin with $P(H) = p$, until 1^{st} head.

let X = # flips until 1^{st} head.

$\therefore X$ is geometric with Param. p

PMF $$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event that 1^{st} head occurs within n flips.

$$\Omega = \{1, 01, 001, 0001, 00001, \ldots\ldots\}$$

$$A = \{1, 01, 001, 0001, \ldots, \underbrace{000\ldots01}_{n}\}$$

$$P(A) = \sum_{k=1}^{n} P_X(k)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^k \qquad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= p \cdot \frac{1-(1-p)^n}{1-(1-p)} = \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{p}}$$

$$= 1-(1-p)^n \quad \checkmark$$

or

$$P(A) = 1 - P(A^c)$$

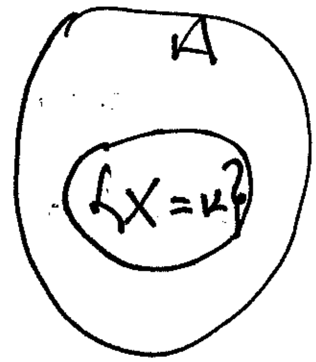
$$= 1 - P(1^{st}\ n \text{ flips all tails})$$

$$= 1-(1-p)^n \quad \checkmark$$

Thus

$$P_{X|A}(k) = \frac{P(\{X=k\} \cap A)}{P(A)} \qquad (k=1,\dots,n)$$

$$= \frac{P(\{X=k\})}{P(A)}$$



and

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1}\cdot p}{1-(1-p)^n} & \text{if } k=1,\dots,n \\ 0 & \text{otherwise} \end{cases}$$

check: $\sum_{k=1}^{n} P_{X|A}(k) = 1$.

Now let Y be another r.v. satisfying

$$P_Y(y) = P(Y=y) > 0$$

for all $y \in \text{range}(Y)$. We can specialize to case

$$A = \{Y = y\}$$

Defn
The conditional PMF of X given Y

$$P_{X|Y}(x|y) = P(X=x \mid Y=y)$$
$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$

We have

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$