CSE 107 1-23-25

Supplemental Lecture

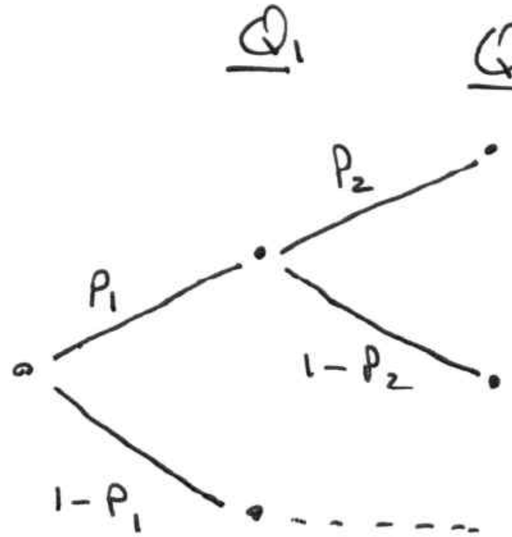
Ex. "Quiz Problem"

A game consists of 2 questions

	Q_1	Q_2
P(correct)	p_1	p_2
Payoff in \$	v_1	v_2

The 1^{st} wrong answer terminates the game. You can answer in either order: (Q_1, Q_2) or (Q_2, Q_1).

Let X be total payoff. Assume order is (Q_1, Q_2). Find PMF of X, and mean of X, $E[X]$.



Prob	X
$p_1 \cdot p_2$	v_1+v_2
$p_1(1-p_2)$	v_1
$1-p_1$	0

$$P_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1+v_2 \\ p_1(1-p_2) & \text{if } x = v_1 \\ 1-p_1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = (v_1+v_2)p_1p_2 + v_1 p_1(1-p_2) + 0\cdot(1-p_1)$$

$$= v_1 p_1 + v_2 p_1 p_2$$

Let Y be the payoff for other order: (Q_2, Q_1). find PMF of Y and $E[Y]$.

swap $1 \leftrightarrow 2$:

$$P_Y(y) = \begin{cases} p_1 p_2 & \text{if } y = v_1 + v_2 \\ (1-p_1)p_2 & \text{if } y = v_2 \\ 1-p_2 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[Y] = v_2 p_2 + v_1 p_1 p_2$$

when should you choose (Q_1, Q_2)?
choose ~~iff~~ $E[X] > E[Y]$, i.e.

$$v_1 p_1 + v_2 p_1 p_2 > v_2 p_2 + v_1 p_1 p_2$$

$$v_1 p_1 - v_1 p_1 p_2 > v_2 p_2 - v_2 p_1 p_2$$

$$v_1 p_1 (1 - p_2) > v_2 p_2 (1 - p_1)$$

$$\frac{v_1 p_1}{1 - p_1} > \frac{v_2 p_2}{1 - p_2}$$

Exercise

Re-do this with 3 questions,
so no $3! = 6$ possible orders.

2.5 Joint PMFs, multiple r.v.s

Defn

Given 2 r.v.s X, Y their <u>joint PMF</u> is

$$p_{X,Y}(x,y) = \underbrace{P(X=x,\ Y=y)}_{\text{shorthand for } P(\{X=x\}\cap\{Y=y\})}$$

with $p_{X,Y}(x,y)$ we can find the Prob. of any event definable in terms of X and Y. For instance

$$A \subseteq \text{range}(X) \times \text{range}(Y)$$

Then

$$P\big((X,Y)\in A\big) = \sum_{(x,y)\in A} p_{X,Y}(x,y)$$

<u>Ex</u>

suppose range$(X) = \{1,2,3,4\}$ and range$(Y) = \{1,2,3\}$. Let the joint PMF be:

$p_X(x)$

		$x=1$	$x=2$	$x=3$	$x=4$	$p_Y(y)$
$p_X(x)$		$4/20$	$5/20$	$6/20$	$5/20$	
Y	3	$1/20$	$1/20$	$2/20$	$1/20$	$5/20$
	2	$2/20$	$3/20$	$3/20$	$1/20$	$9/20$
	1	$1/20$	$1/20$	$1/20$	$3/20$	$6/20$

x

$p_{X,Y}(x,y)$

what is $P(Y=3)$. since

$$\{X=1\}\cup\{X=2\}\cup\{X=3\}\cup\{X=4\}=\Omega$$

is a disjoint union, by the total Probability theorem, we have

$$P(Y=3)=P(X=1,Y=3)+P(X=2,Y=3)+P(X=3,Y=3)+P(X=4,Y=3)$$

$$=\sum_{x=1}^{4}P_{X,Y}(x,3) = 5/20$$

In general

$$\boxed{P_X(x)=\sum_y P_{X,Y}(x,y)} \quad \text{(marginal)}$$

$$\boxed{P_Y(y)=\sum_x P_{X,Y}(x,y)} \quad \text{(marginal)}$$

Functions of multiple r.v.s

Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and let

$$Z = g(X, Y)$$

Then

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x,y)$$

By the total Prob. theorem.

The expected value rule generalizes to

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)}$$

Exercise Prove this.

Ex. $E[aX+bY+c] = aE[X] + bE[Y] + c$ ✓

Proof:

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c)\, p_{X,Y}(x,y)$$

$$= a \sum_x \sum_y x\, p_{X,Y}(x,y) + b \sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$+ c \sum_x \sum_y p_{X,Y}(x,y)$$

$$= a \sum_x x \left(\sum_y p_{X,Y}(x,y) \right) + b \sum_y y \left(\sum_x p_{X,Y}(x,y) \right) + c$$

$$= a \sum_x x\, p_X(x) + b \sum_y y\, p_Y(y) + c$$

$$= a E[X] + b E[Y] + c$$

■

Ex Previous

$$E[X+3Y]=E[X]+3E[Y]$$

$$=\left(1\cdot\frac{4}{20}+2\cdot\frac{5}{20}+3\cdot\frac{6}{20}+4\cdot\frac{5}{50}\right)$$

$$+3\left(1\cdot\frac{6}{20}+2\cdot\frac{9}{20}+3\cdot\frac{5}{20}\right)$$

$$=\boxed{\frac{169}{20}}$$

More than 2 random Variables

3 Variables: X, Y, Z

$$P_{X,Y,Z}(x,y,z)=P(X=x, Y=y, Z=z)$$

we have 3 - 2 var. marginal PMFs

$$P_{X,Y}(x,y) = \sum_{z} P_{X,Y,Z}(x,y,z)$$

$$P_{X,Z}(x,z) = \sum_{y} P_{X,Y,Z}(x,y,z)$$

$$P_{Y,Z}(y,z) = \sum_{x} P_{X,Y,Z}(x,y,z)$$

also have 3 - 1 var. marginal PMFs ...

Now suppose we have n variables

$$X_1, X_2, X_3, \ldots, X_n$$

then the generalized expected value rule gives.

$$E[a_1X_1 + a_2X_2 + \cdots + a_nX_n + b]$$

$$= a_1E[X_1] + \cdots\cdots + a_nE[X_n] + b$$

<u>The mean of the Binomial r.v. with</u>

<u>Parameters n, p.</u>

Recall

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (0 \le k \le n)$$

To compute

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Let

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i $(1 \le i \le n)$ is Bernoulli with param. p. so

$$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$$

Thus

$$E[X] = E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= \underbrace{p + \cdots + p}_{n}$$

$$= np$$

will show later that:

$$\mathrm{Var}(X) = np(1-p).$$