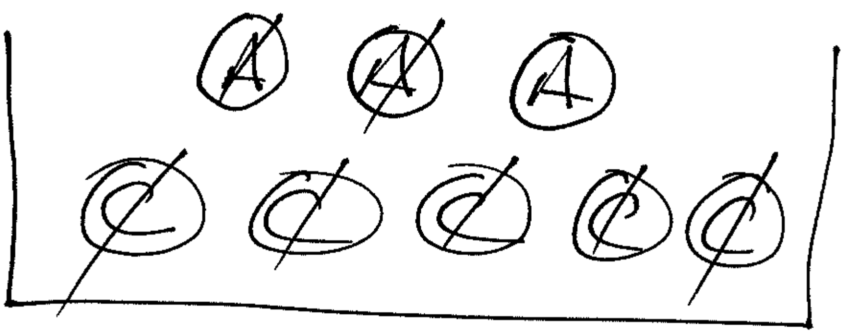
Remarks on lab 2

Suppose $a = 3$, $c = 5$



① start: ~~C~~ ~~C~~ A ← replace

② start: ~~A~~ ~~A~~ C ← replace

③ start: ~~C~~ ~~C~~ ~~C~~ A ← replace

④ start: A ← last draw is Azure.

Ex. Bernoulli r.v. with Param. p

$$P_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$$

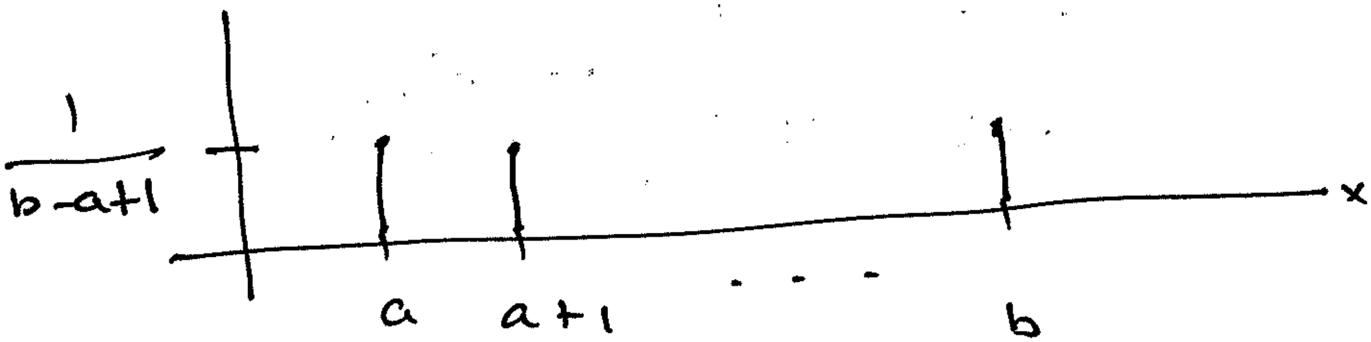
$$E[X] = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= p - p^2 = \boxed{p(1-p)}$$

Ex Discrete Uniform on $\{a, \dots, b\}$



$$p_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \dots, b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{k=a}^{b} k \cdot \left(\frac{1}{b-a+1}\right)$$

$$= \frac{1}{b-a+1} \cdot \sum_{k=a}^{b} k$$

$$= \left(\frac{1}{b-a+1}\right)\left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k\right)$$

$$= \frac{1}{b-a+1} \cdot \left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2} \right)$$

$$= \frac{1}{2} \cdot \frac{1}{b-a+1} \cdot \left(b^2 + b - a^2 + a \right)$$

$$= \frac{1}{2} \frac{1}{b-a+1} \left(\underbrace{(b^2-a^2)}_{(b-a)(b+a)} + (b+a) \right)$$

$$= \frac{a+b}{2} \cdot \frac{1}{\cancel{b-a+1}} \cdot \cancel{(b-a+1)} = \frac{a+b}{2}$$

To get $Var(X)$, first set

$Y = X + (1-a)$, so

$$Var(Y) = Var(X + (1-a)) = Var(X)$$

Let $n = b-a+1$, and note Y is uniform on $\{1, 2, \ldots, n\}$

since

$$X = a \Rightarrow Y = a + (1-a) = 1$$

$$X = b \Rightarrow Y = b-a+1 = n$$

Thus $E[Y] = \frac{n+1}{2}$, and

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$\therefore \operatorname{Var}(X) = \operatorname{Var}(Y) = E[Y^2] - (E[X])^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$= \text{-----} \text{ exercise } \text{----}$

$$= \frac{(n+1)(n-1)}{12}$$

$$= \frac{(b-a)(b-a+2)}{12}.$$

Ex Poisson r.v. with Param $\lambda > 0$.

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \text{if } k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

we will show

(1) $E[X] = \lambda$

(2) $E[X^2] = \lambda^2 + \lambda$

<u>Proof of (1)</u>

$$E[X] = \sum_{k=\cancel{0}1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \qquad (*)$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot e^{-\lambda} \cdot \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \begin{cases} \text{replace} \\ k \leftarrow k+1 \end{cases}$$

$$= \lambda \cdot \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \boxed{\lambda}$$

<u>Proof of (2)</u>

$$E[X^2] = \sum_{k=\cancel{0}\,1}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=0}^{\infty} (k+1) \cdot \frac{\lambda^k}{k!} \quad \begin{cases} \text{replace} \\ k \leftarrow k+1 \end{cases}$$

$$= \lambda e^{-\lambda} \left(\sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda \Big(\underbrace{e^{-\lambda} \cdot \sum_{k=\cancel{0}\,1}^{\infty} k \cdot \frac{\lambda^k}{k!}}_{\lambda} + \underbrace{e^{-\lambda} \cdot e^{\lambda}}_{1} \Big)$$

$$= \lambda(\lambda + 1) = \lambda^2 + \lambda .$$

so

$$Var(X) = (\lambda^2 + \lambda) - \lambda^2$$

$$= \boxed{\lambda}$$