CSE 107 1-20-25

Supplemental Lecture

2.4 Expectation, Mean and Variance

Defn

The expected value of a r.v. X (also mean, expectation or average) is

$$E[X] = \sum_x x \, p_X(x)$$

where sum is over all $x \in \text{range}(X)$

In Physics this is center of mass.

Ex Previous ex.

$$E[X] = (-1)(.2) + (1)\cdot(.3) + (2)(.5) = \boxed{1.1}$$

$$E[Y] = E[X^2] = (1)(.5) + (4)(.5) = \boxed{2.5}$$

note: $E[X^2] \neq E[X]^2$

Expected Value rule

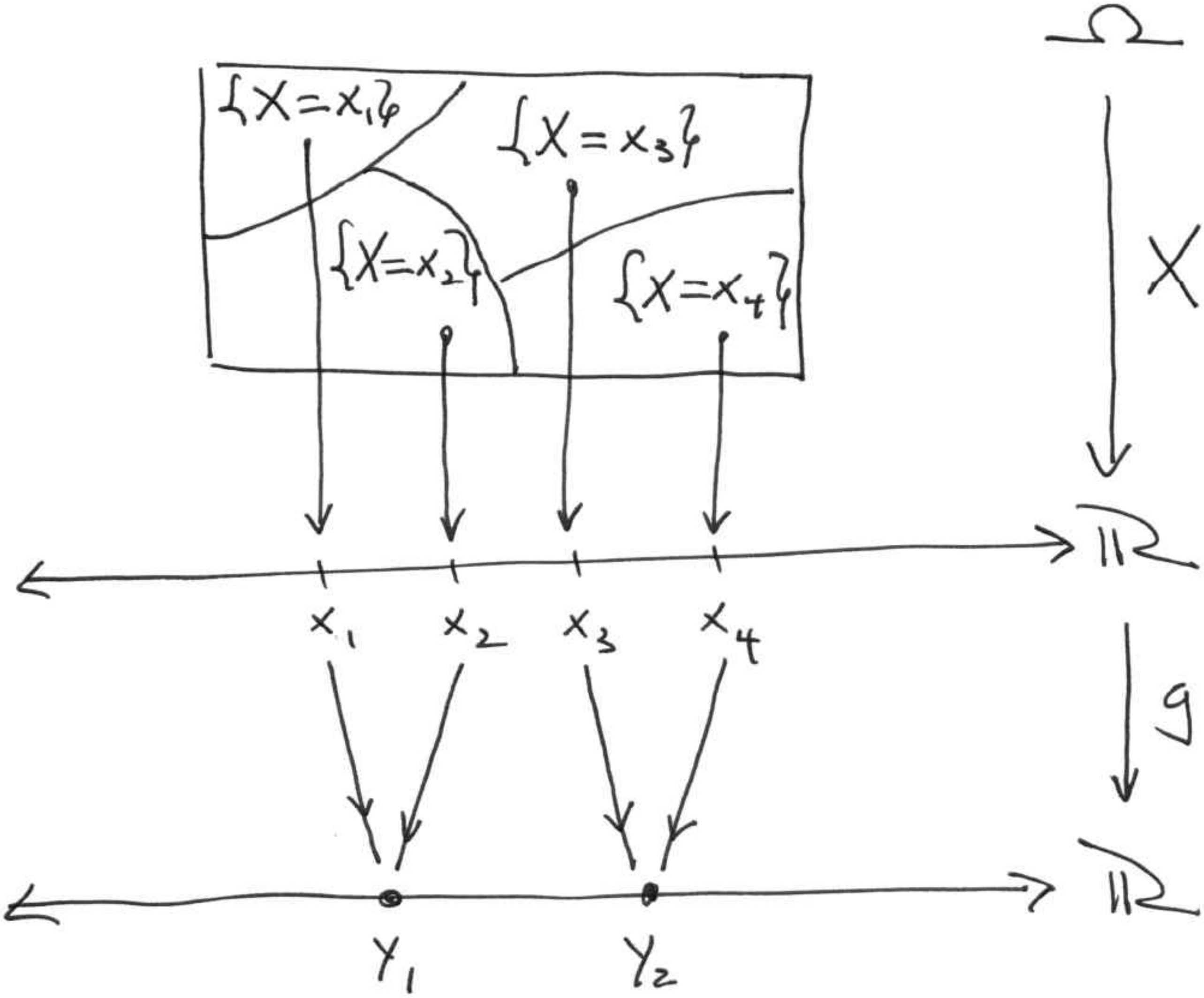
$$E[g(X)] = \sum_x g(x)\cdot p_X(x)$$

Picture:

suppose $\text{range}(X) = \{x_1, x_2, x_3, x_4\}$ and

suppose $g: \mathbb{R} \to \mathbb{R}$ is such that

$$g(x_1) = g(x_2) = y_1 \text{ and } g(x_3) = g(x_4) = y_2$$



$$E[Y] = y_1 P_Y(y_1) + y_2 P_Y(y_2)$$

$$= y_1 \big(P_X(x_1) + P_X(x_2)\big) + y_2 \big(P_X(x_3) + P_X(x_4)\big)$$

$$= \big(y_1 P_X(x_1) + y_1 P_X(x_2)\big) + \big(y_2 P_X(x_3) + y_2 P_X(x_4)\big)$$

$$= g(x_1) P_X(x_1) + g(x_2) P_X(x_2) + g(x_3) P_X(x_3) + g(x_4) P_X(x_4)$$

$$= \sum_x g(x) P_X(x)$$

Proof

observe

$$\{Y=y\} = \{\omega \mid Y(\omega)=y\}$$

$$= \{\omega \mid g(X(\omega))=y\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{\omega \mid X(\omega)=x\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{X=x\}$$ [disjoint union

$$\therefore \; P_Y(y) = P(Y=y)$$

$$= \sum_{\{x \mid g(x)=y\}} P(X=x)$$

$$= \sum_{\{x \mid g(x)=y\}} P_X(x)$$

now

$$E[Y] = \sum_y y \, p_Y(y)$$

$$= \sum_y y \cdot \sum_{\{x \mid g(x) = y\}} p_X(x)$$

$$= \sum_y \sum_{\{x \mid g(x) = y\}} y \, p_X(x)$$

$$= \sum_y \sum_{\{x \mid g(x) = y\}} g(x) \, p_X(x)$$

$$= \sum_x g(x) \, p_X(x)$$

$\blacksquare$

Ex (Previous)

$$E[X^2] = (-1)^2(.2) + (1)^2(.3) + (2)^2(.5)$$

$$= .2 + .3 + 4(.5) = \boxed{2.5}$$

Ex $Y = aX + b$.

$$E[Y] = E[aX + b]$$

$$= \sum_x (ax+b) p_X(x)$$

$$= \sum_x (ax\, p_X(x) + b\, p_X(x))$$

$$= a \cdot \sum_x x\, p_X(x) + b \cdot \sum_x p_X(x)$$

$$= a E[X] + b$$

If $b=0$: $E[aX] = aE[X]$.

If $a=0$: $E[b] = b$.

Defn

The n^{th} moment of X $(n=0,1,2,\ldots)$ is $E[X^n]$.

Thus $E[X]$ is the 1^{st} moment and $E[X^2]$ is " 2^{nd} "

Defn

The Variance of X is the mean of the r.v. $(X-E[X])^2$.

note: since $(X-E[X])^2$ is always non-negative, we have

$$Var(X) = E\left[(X-E[X])^2\right]$$

is necessarily non-negative.

Defn

The standard deviation of X is

$$\sigma_X = \sqrt{Var(X)}$$

Both $Var(X)$ and σ_X are measures of the dispersion of X.

Ex (Previous)

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & x = 1 \\ .5 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Previously:

$E[X] = 1.1$ 1^{st} moment

$E[X^2] = 2.5$ 2^{nd} moment

By expected value rule

$$Var(X) = E[(X - \underbrace{E[X]}_{1.1})^2]$$

$$= \sum_{x} (x - (1.1))^2 p_X(x)$$

$$= (-1-(1.1))^2 \cdot (.2) + (1-(1.1))^2 (.3) + (2-(1.1))^2 \cdot (.5)$$

$$= \boxed{1.29}$$

and

$$\sigma_X = \boxed{1.1358}$$

There's an even simpler formula for $Var(X)$.

<u>Variance in terms of moments</u>

$$Var(X) = E[X^2] - (E[X])^2$$

<u>notation</u>: $\mu = E[X]$.

<u>Proof</u>

$$Var(X) = E[(X-\mu)^2]$$

$$= \sum_x (x-\mu)^2 p_X(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\sum_x \mu x p_X(x) + \sum_x \mu^2 p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\mu \sum_x x p_X(x) + \mu^2 \sum_x p_X(x)$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

$\blacksquare$

Ex (previous)

we had $E[X] = 1.1$, $E[X^2] = 2.5$

Thus

$$Var(X) = 2.5 - (1.1)^2 = \boxed{1.29}$$

Ex Recall if $Y = aX + b$, then

$$E[Y] = E[aX+b] = aE[X] + b.$$

find 2nd moment of Y:

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 p_X(x)$$

$$= \sum_x \left(a^2x^2 p_X(x) + 2abx\, p_X(x) + b^2 p_X(x)\right)$$

$$= a^2 \sum_x x^2 p_X(x) + 2ab \sum_x x\, p_X(x) + b^2 \sum_x p_X(x)$$

$$= a^2 E[X^2] + 2ab\, E[X] + b^2$$

now we have

$$Var(Y) = Var(aX + b)$$

$$= E[(aX+b)^2] - E[aX+b]^2$$

$$= \left(a^2 E[X^2] + 2ab\, E[X] + b^2\right) - \left(a E[X] + b\right)^2$$

$$= a^2 E[X^2] + \cancel{2ab\, E[X]} + \cancel{b^2} - \left(a^2 E[X]^2 + \cancel{2ab\, E[X]} + \cancel{b^2}\right)$$

$$= a^2 \left(E[X^2] - E[X]^2 \right)$$

$$= a^2\, Var(X)$$

Summary

- $E[aX+b] = a\,E[X] + b$
- $Var(aX+b) = a^2\,Var(X)$.