ese 107 1-17-25

The Poisson Random Variable

PMF: $P_X(k) = e^{-\lambda} \cdot \dfrac{\lambda^k}{k!} \quad (k=0,1,2,\ldots)$

note we specified the PMF of X not $X: \Omega \to \mathbb{R}$. observe

$$\sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1. \checkmark$$

$\lambda > 0$ is sometimes called the arrival rate.

Poisson r.v. is the continuous time limit of the Binomial.

$$\boxed{e^{-\lambda} \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k}} \quad \text{for } k = 0, 1, \dots, n$$

where $\lambda = np$, n is large, p is small.

more precisely, let $\lambda > 0$ be constant and $p = \frac{\lambda}{n}$. Then

$$\lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$$

<u>Ex</u> let $n = 200$, $\lambda = 2$, $p = \frac{\lambda}{n} = .01$.

The prob. of $k = 4$ successes in 200 ind. trials with prob. $p = .01$ of success is

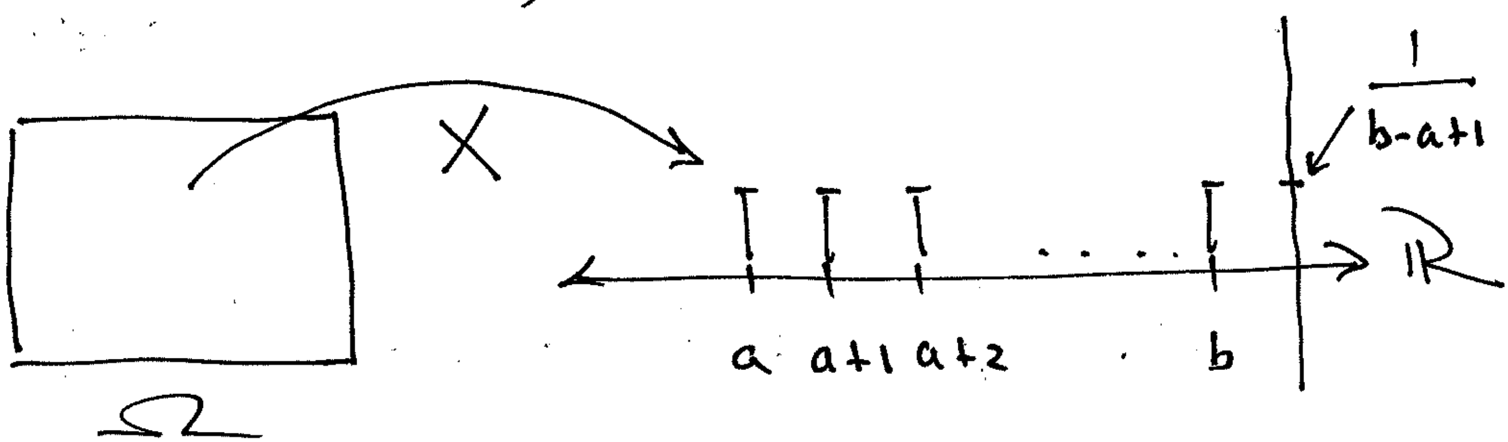
$$\binom{200}{4}(.01)^4(.99)^{196} = .0902197$$

This is well approximated by

$$e^{-2}\cdot\frac{2^4}{4!} = .0902235$$

Discrete Uniform Random Variable

let $a, b \in \mathbb{Z}$, $a \leq b$



$$\text{range}(X) = \{a, a+1, \ldots, b\}$$

$$|\{a, a+1, \ldots, b\}| = b - a + 1$$

$$P_X(k) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

2.3 Functions of a random Variable

Given $X:\Omega\to\mathbb{R}$ an r.v.,

we can transform it with a

function

$$g:\mathbb{R}\to\mathbb{R}$$

using composition: $Y=g\circ X=g(X)$.

We have $Y=g\circ X:\Omega\to\mathbb{R}$,

so Y is itself a random

variable

Ex. Suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$. Find the PMF of Y.

$$\begin{aligned} P_Y(y) &= P(Y = y) \\ &= P(X^2 = y) \\ &= P(X = \pm\sqrt{y}) \end{aligned}$$

$$= P(X = \sqrt{y} \text{ or } X = -\sqrt{y})$$

$$= P(\{X = \sqrt{y}\} \cup \{X = -\sqrt{y}\})$$

$$= P(X = \sqrt{y}) + P(X = -\sqrt{y})$$

$$= \begin{cases} \cancel{.2 \quad \text{if } \sqrt{y} = -1} \\ .3 \quad \sqrt{y} = 1 \\ .5 \quad \sqrt{y} = 2 \end{cases} + \begin{cases} .2 \quad \text{if } -\sqrt{y} = -1 \\ \cancel{.3 \quad -\sqrt{y} = 1} \\ \cancel{.5 \quad -\sqrt{y} = 2} \end{cases}$$

$$= \begin{cases} .3 + .2 & \text{if } \sqrt{y} = 1 \\ .5 & \text{if } \sqrt{y} = 2 \end{cases}$$

$$= \begin{cases} .5 & \text{if } y = 1 \\ .5 & \text{if } y = 4 \\ 0 & \text{otherwise} \end{cases}$$