CSE 107 1-16-24

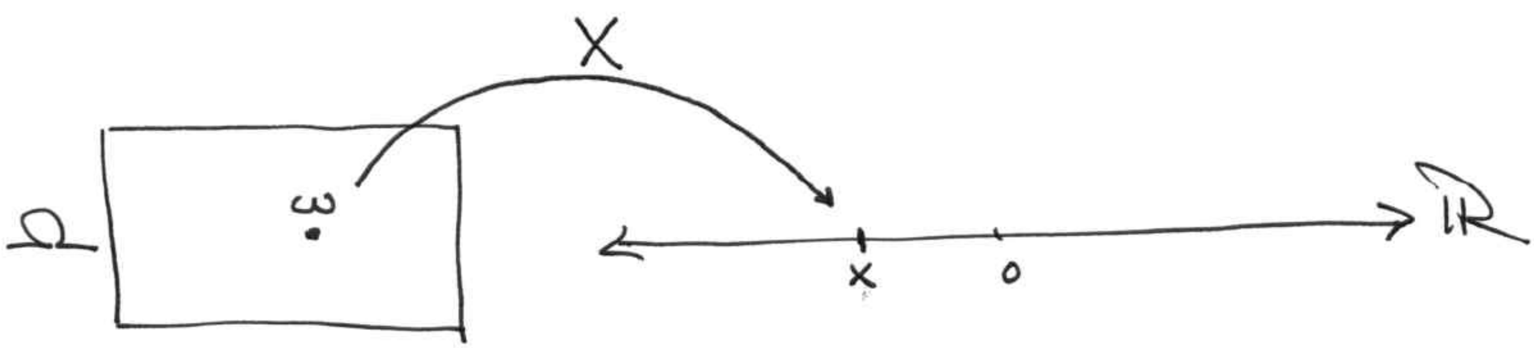
Supplemental Lecture

2.1 Random Variables

Defn

A random variable is a function

$$X : \Omega \to \mathbb{R}$$



notation

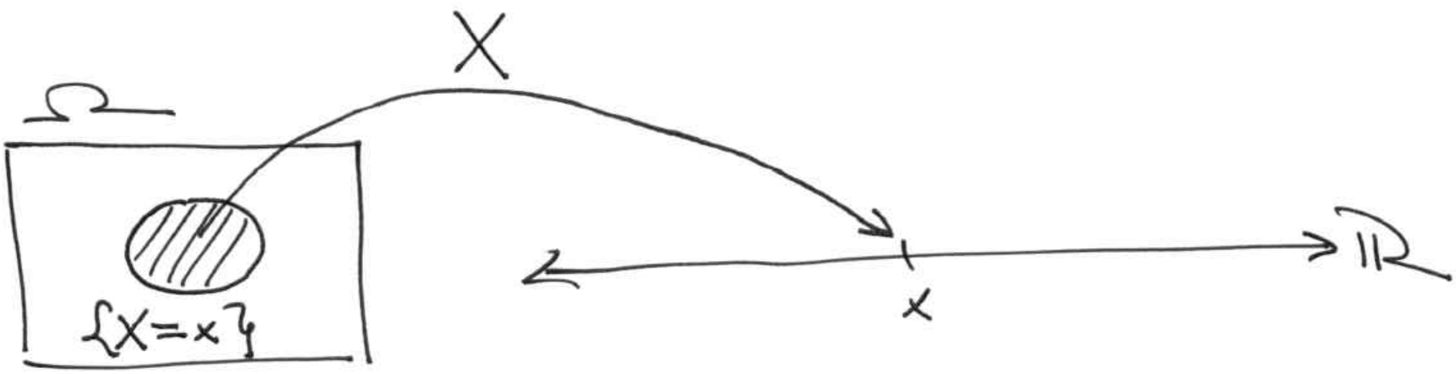
- We always use capitol letters for r.v.'s, and lower case letters for values

$$X(\omega) = x$$

- We denote by $\{X=x\}$ the event

$$\{X=x\}=\{\omega\in\Omega \mid X(\omega)=x\}\subseteq\Omega$$

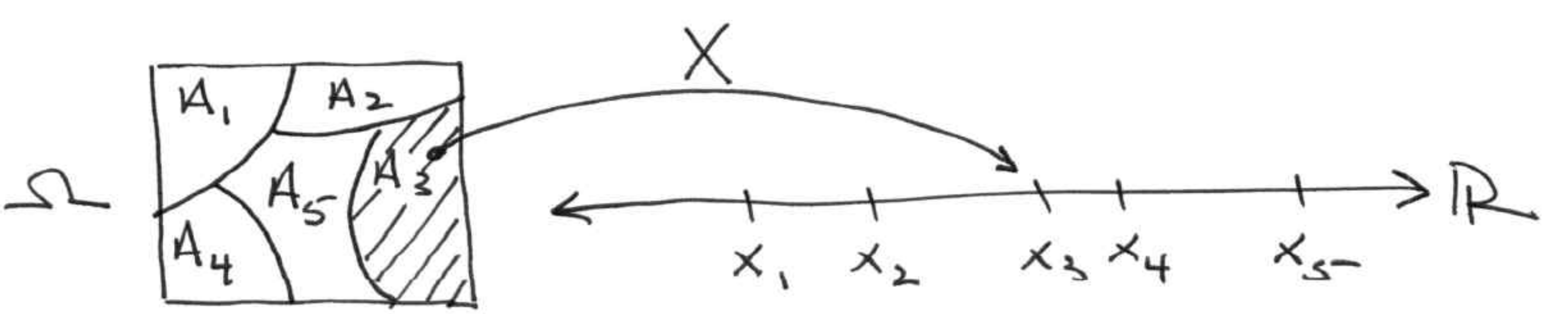
called the <u>preimage</u> of x under X.



- X is called <u>discrete</u> if its <u>range</u>

$$\text{range}(X)=\{X(\omega)\mid\omega\in\Omega\}\subseteq\mathbb{R}$$

is a discrete subset of $\mathbb{R}$.



$A_i=\{X=x_i\}$

- X is called <u>continuous</u> ......, <u>chap3</u>

- some r.v.s are neither discrete nor continuous.

<u>Ex.</u> (discrete)

Roll 2 fair indep. dice, let

$$X((i,j)) = i+j$$

i.e. sum of top 2 faces.

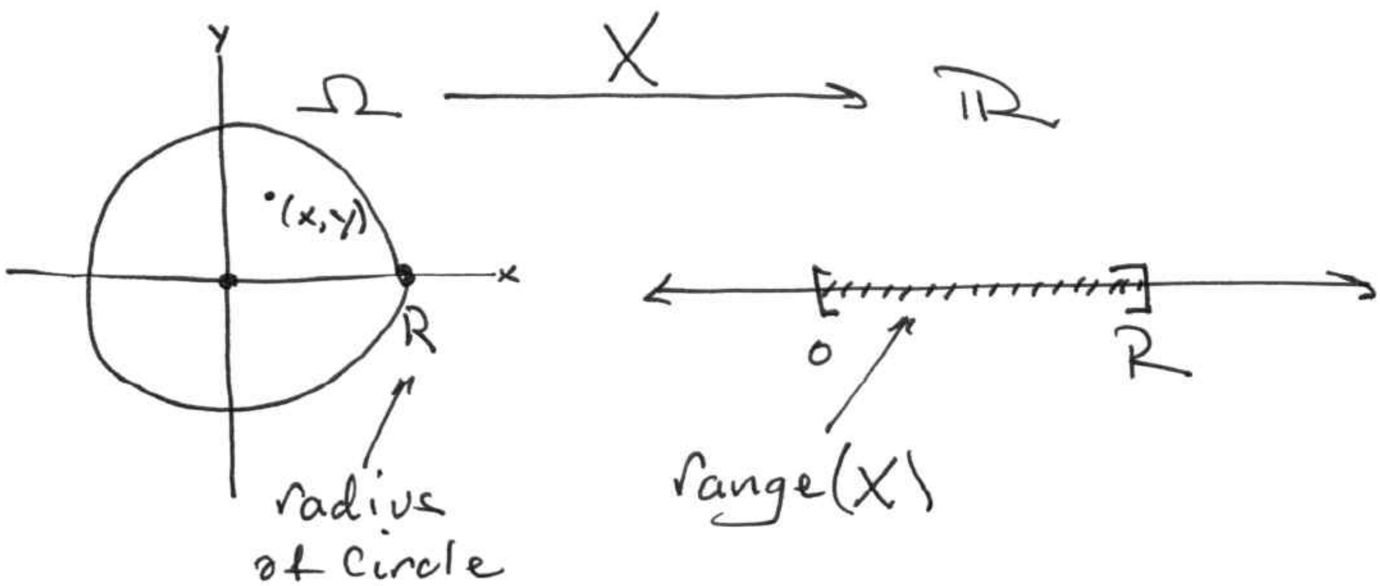
$$\Omega \begin{Bmatrix} 11 & 12 & \cdots & 16 \\ 21 & 22 & \cdots & 26 \\ \vdots & & & \\ 61 & 62 & \cdots & 66 \end{Bmatrix} \xrightarrow{X} \underbrace{\{2,3,4,5,6,7,8,9,10,11,12\}}_{\text{range}(X)} \subseteq \mathbb{R}$$

Ex. (continuous)

Throw a dart at circular target centered at (0,0). Let

$$X\big((x,y)\big) = \sqrt{x^2+y^2}$$

the distance of dart at (x,y) from $(0,0)$



2.2 Probability mass functions

Defn

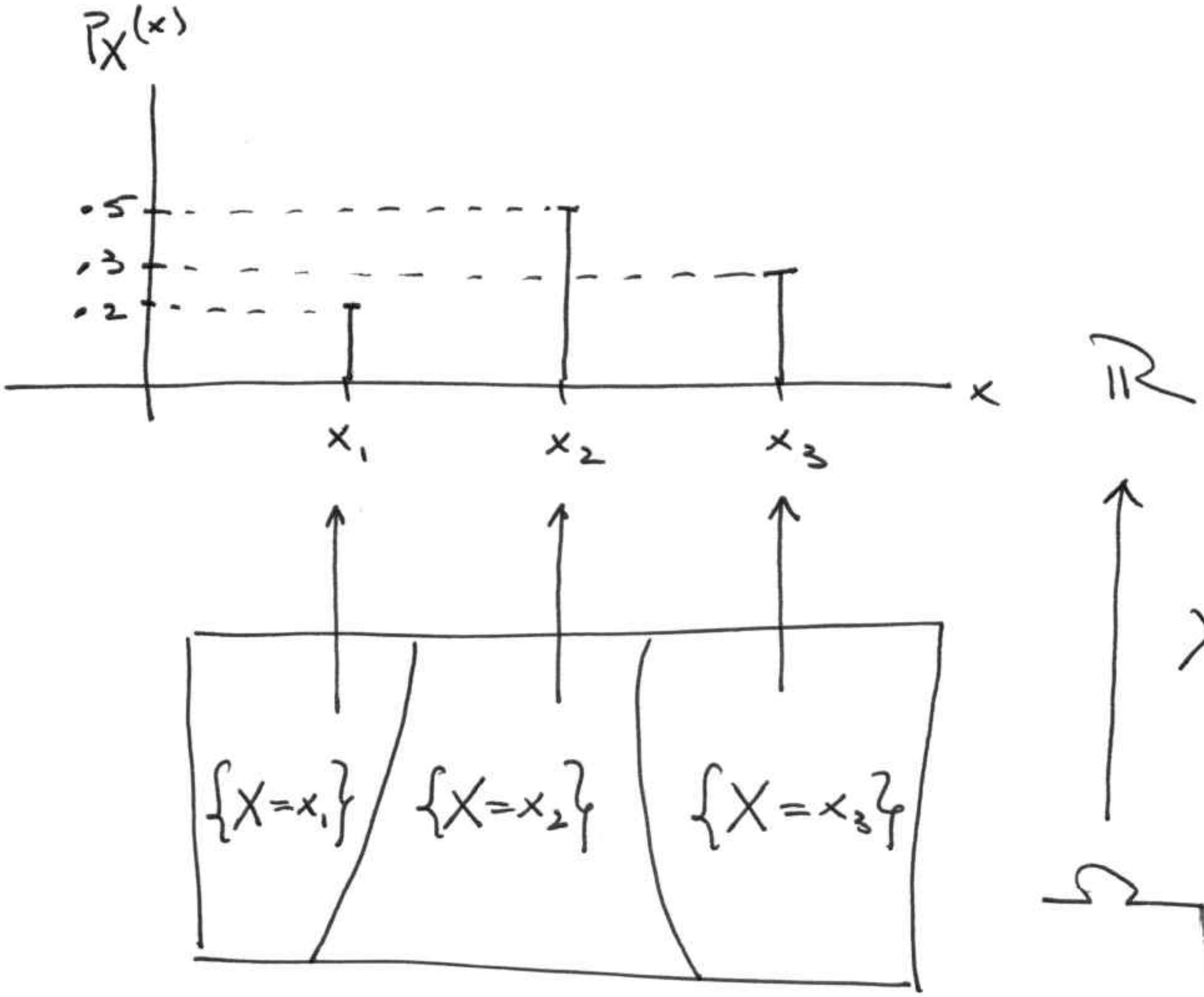
Let X be a discrete r.v. The Probability mass function (PMF) of X is

$$p_X(x) = \mathbb{P}(\{X=x\}) = \mathbb{P}(X=x)$$

observe $p_X : \mathbb{R} \to [0,1]$. Also, we necessarily have

$$\sum_{x \in \text{range}(X)} p_X(x) = 1$$

Ex



so

$$P_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

note: $\sum_x P_X(x) = P_X(x_1) + P_X(x_2) + P_X(x_3)$

$= .2 + .5 + .3 = 1$ ✓

The Bernoulli Random Variable

Perform 1 Bernoulli trial, define

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is success} \\ 0 & \text{if } \omega \text{ is failure} \end{cases}$$

It's PMF is

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

where $P(\text{success}) = p$, and $P(\text{failure}) = 1-p$.

We call p the <u>parameter</u> of the Bernoulli r.v.

The Binomial Random Variable

Perform a sequence of n independent Bernoulli trials. ($\Omega = \{$bit strs. of len. $n\}$)

$$X(\omega) = \text{\# of successes in } n \text{ trials}$$

The PMF of X is the Binomial Probabilities of sec. 1.5.

$$p_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \dots, n$. Actually

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k = 0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

check:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \left(p + (1-p)\right)^n = 1^n = 1 \quad \checkmark$$

The Geometric Random Variable

Perform independent Bernoulli trials (with param. p) until first success.

$\Omega = \{1, 01, 001, 0001, 00001, \ldots\ldots\}$. Let

$$X(\omega) = \#\text{ of trials until } 1^{st} \text{ success} = \text{length}(\omega)$$

so $P_X(k) = P(X=x)$ is the probability of a sequence of $k-1$ failures, followed by 1 success. Thus $P_X(k) = (1-p)^{k-1} \cdot p$, i.e.

$$P_X(k) = \begin{cases} p(1-p)^{k-1} & \text{if } k = 1, 2, 3, \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

Note:

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p \sum_{k=0}^{\infty} (1-p)^k = p \cdot \frac{1}{1-(1-p)} = 1$$