CSE 107 1-13-25

HW1 #2

Players: 1, 2, 3

strengths: $1 < 2 < 3$

$A = \{\text{Win against } 1\}$, $P(A) = p_1$

$B = \{\text{" " } 2\}$, $P(B) = p_2$

$C = \{\text{" " } 3\}$, $P(C) = p_3$

Know: $p_1 > p_2 > p_3$

suppose we play in order $(1, 2, 3)$

let $E = \{\text{Win Tournament}\} = (A \cap B) \cup (B \cap C)$

$$P(A \cap B) = P(A) \cdot \underbrace{P(B|A)}_{\substack{\text{Assume} \\ = P(B)}} = P(A) \cdot P(B) = P_1 \cdot P_2$$

$$P(B \cap C) = P(B) \cdot \underbrace{P(C|B)}_{P(C)} = P(B) \cdot P(C) = P_2 \cdot P_3$$

$$P(A \cap B \cap C) = P(A) \cdot \underbrace{P(B|A)}_{P(B)} \cdot \underbrace{P(C|A \cap B)}_{P(C)} = P_1 \cdot P_2 \cdot P_3$$

1.5 Independence

It's possible that B occurs tells you nothing as to whether A occurs.

(*) $$P(A|B) = P(A)$$

We say A is independent of B.
so

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

and

(**) $$P(A \cap B) = P(A) \cdot P(B)$$

<u>Defn</u>

We say $A, B \subseteq \Omega$ are <u>independent</u> iff

$$P(A \cap B) = P(A) \cdot P(B)$$

<u>EX</u> throw two <u>fair</u> dice. let

$$A_i = \{1^{st} \text{ die is } i\} \quad (1 \leq i \leq 6)$$

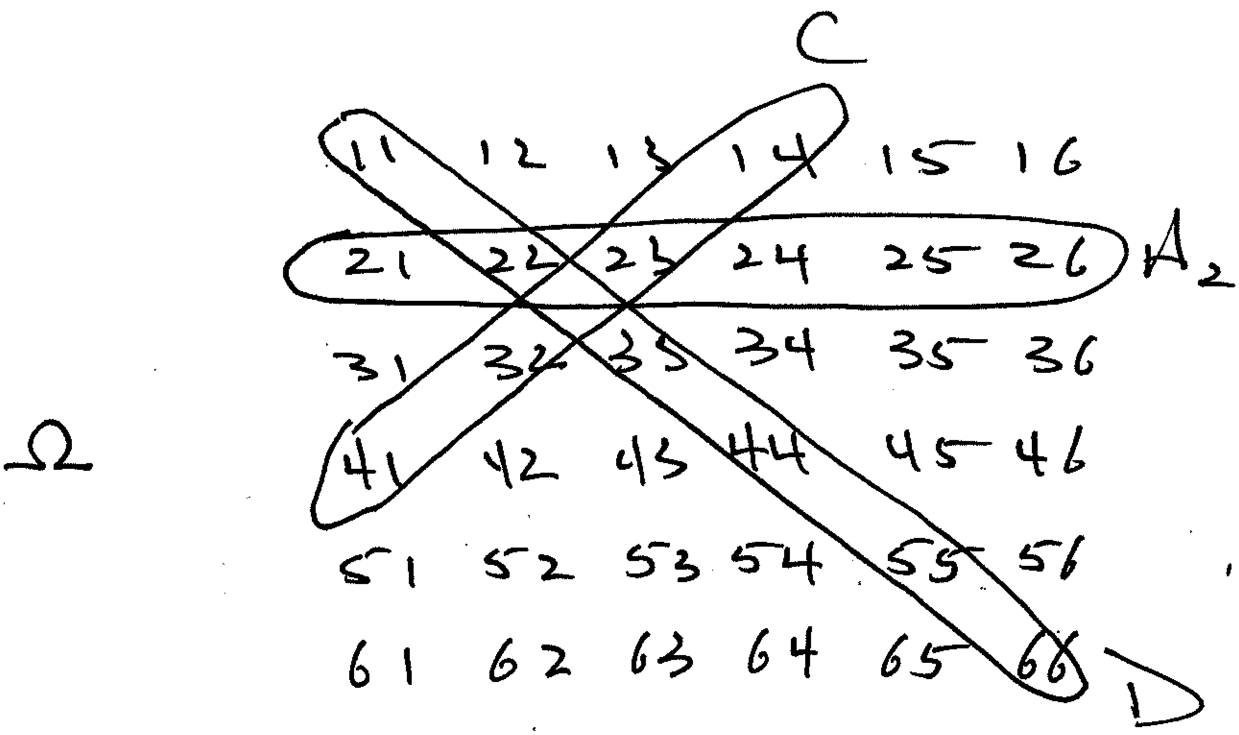
$$B_j = \{2^{nd} \text{ die is } j\} \quad (1 \leq j \leq 6)$$

Assume A_i, B_j are <u>independent</u>, for all i, j.

Then

$$P(ij) = P(A_i \cap B_j)$$
$$= P(A_i) \cdot P(B_j) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Recall



Let $C = \{\text{sum} = 5\} = \{41, 32, 23, 14\}$

$\therefore P(C) = \frac{4}{36} = \frac{1}{9}.$

Is C independent of A_2 ?

$$A_2 \cap C = \{23\}, \quad P(A_2 \cap C) = \frac{1}{36}$$

$$P(A_2) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54} \neq \frac{1}{36}$$

$\therefore$ $P(A_2 \cap C) \neq P(A_2) \cdot P(C)$

$\therefore$ A_2 and C are <u>not ind</u>.

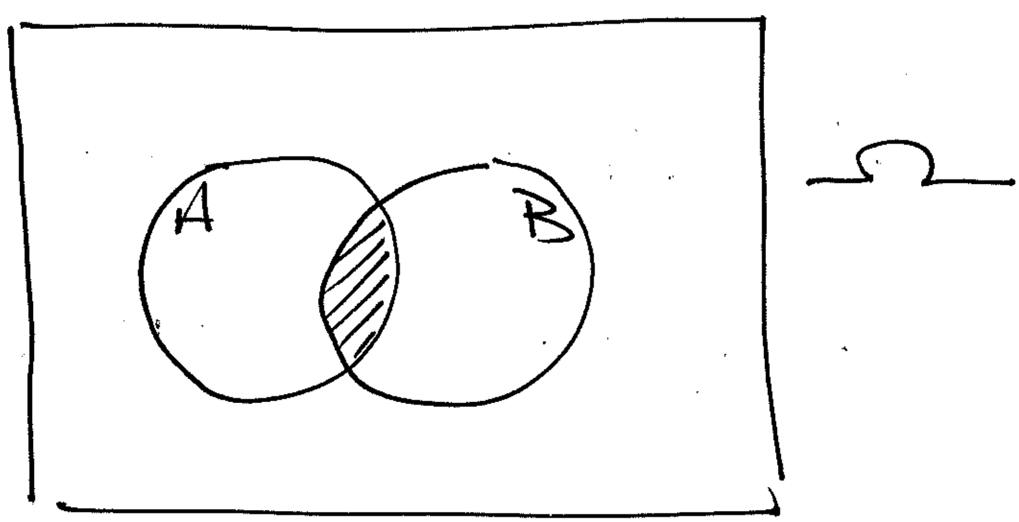
Let $D = \{\text{doubles}\}$, $P(D) = \frac{6}{36} = \frac{1}{6}$

$D \cap A_2 = \{22\}$ so $P(D \cap A_2) = \frac{1}{36}$

$$P(D) \cdot P(A_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(D \cap A_2)$$

$\therefore$ D and A_2 <u>are</u> <u>independent</u>.

If Prob. are normalized area in Ω, then indep. of A and B means



$$\frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(\Omega)}$$

i.e. $$\frac{\text{area}(A \cap B)}{\text{area}(B)} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

Exercise

Let $A, B \subseteq \Omega$. Prove that the following statements are equivalent.

(a) A and B are independent

(b) A^c " B " "

(c) A " B^c " "

(d) A^c " B^c " "

Conditional independence

Recall if $P(C) > 0$, then

$$P(\cdot \mid C)$$

is a valid Prob. law. (i.e. axioms are satisfied.)

Defn

We say A, B are conditionally independent given C iff

$$P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

Interesting equivalent Defn.

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}$$

mult. rule

$$= \frac{\cancel{P(C)} \cdot P(B \mid C) \cdot P(A \mid B \cap C)}{\cancel{P(C)}}$$

$$= P(B \mid C) \cdot P(A \mid B \cap C)$$

If A, B are cond. ind. given C:

$$\cancel{P(B \mid C)} \cdot P(A \mid B \cap C) = P(A \mid C) \cdot \cancel{P(B \mid C)}$$

$$\therefore \boxed{P(A \mid B \cap C) = P(A \mid C)}$$

alternate defn of cond. indep.